# THE BRAUER-MANIN OBSTRUCTION FOR NONISOTRIVIAL CURVES OVER GLOBAL FUNCTION FIELDS

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Abstract. We prove that the set of rational points on a nonisotrivial curves of genus at least 2 over a global function field is equal to the set of adelic points cut out by the Brauer-Manin obstruction.

## 1. Introduction

Let $X/K$ be a smooth projective and geometrically irreducible curve of genus at least 2 over a global field $K$ of characteristic $p > 0$. We prove that if $X$ is not isotrivial, then the Brauer-Manin obstruction cuts out exactly the set of rational points on $X$.

**Theorem 1.1.** *Let $X/K$ be a smooth projective curve of genus at least 2 over a global function field $K$. If $X$ is not isotrivial, then $X(\mathbb{A}_K)^{\mathrm{Br}} = X(K)$.*

We refer the reader to [PV10] for the definition of the Brauer-Manin obstruction and the relevant background in this context. Theorem 1.1 is proved in that paper for $X$ contained in an abelian variety $A$ such that $A(K^{\mathrm{sep}})[p^\infty]$ is finite and no geometric isogeny factor of $A$ is isotrivial. That result holds more generally for any 'coset free' subvariety of such an abelian variety. We remove the hypotheses on an abelian variety containing $X$, but our proof does not immediately extend to higher dimensional subvarieties of abelian varieties.

As in [PV10] our results are a consequence of related results concerning adelic intersections whose connection to the Brauer-Manin obstruction was first observed in [Sch99] for curves over number fields.

**Theorem 1.2.** *Suppose $X \subset A$ is a proper smooth curve of genus at least 2 contained in an abelian variety $A$ over a global field $K$ of characteristic $p > 0$. Then $X(\mathbb{A}_k) \cap \overline{A(K)} = X(K)$, where $\overline{A(K)}$ denotes the topological closure of $A(K)$ in $A(\mathbb{A}_K)$.*

We follow the strategy of the proof in [PV10], but there are two new ingredients allowing us to remove all hypotheses on an abelian variety containing $X$. The first, appearing as Proposition 2.3 below, is based on ideas in the proof of the Mordell-Lang Conjecture appearing in [AV92, Vol91]. This replaces the input in [PV10] from [Hru96] which relies heavily on model theory and requires assumptions on the Jacobian of $X$. The second new ingredient is an isogeny constructed by Rössler in the appendix to this paper. We use this instead of multiplication by $p$ in some of the arguments appearing in [PV10] to prove Proposition 3.1. This removes the need for the hypothesis on $A(K^{\mathrm{sep}})[p^\infty]$ in [PV10, Proposition 5.3] and elsewhere.

The theorems above are expected to hold (in a slightly modified form) for any closed subvariety of an abelian variety over a global field. This was originally posed as a question in the case of curves over number fields by Scharaschkin [Sch99] and, independently, by Skorobogatov [Sko01]. It was later stated as a conjecture for curves over number fields in

[Poo06] and [Sto07]. The number field case has seen little progress and remains wide open. Building on [PV10], this paper settles the function field analogue of these conjectures for nonisotrivial curves of genus $\geq 2$. Some partial results toward the conjecture in the isotrivial case are given in [CV22, CV23], but this case too remains open.

## 2. Zariski dense adelic points surviving $p^\infty$-descent

In this section we assume $X \subset A$ is a proper smooth curve of genus $\geq 2$ contained in an abelian variety $A$ over $K$.

**Definition 2.1.** *Let $N \geq 1$ be an integer. An $N$-covering of a subvariety $X \subset A$ of an abelian variety $A$ over $K$ is an fppf-torsor $Y \to X$ under the $N$-torsion subgroup scheme $A[N]$ such that the base change of $Y \to X$ to $K^{\mathrm{sep}}$ is isomorphic to the pull back of multiplication by $N$ on $A$. An adelic point on $X$ is said to survive $N$-descent if it lifts to an adelic point on some $N$-covering of $X$.*

**Definition 2.2.** *An adelic point $(P_v)_v \in X(\mathbb{A}_K)$ is called Zariski dense if for any proper closed subvariety $Y \subsetneq X$, there exists $v$ such that $P_v \notin Y$.*

**Proposition 2.3.** *Suppose $X \subset A$ is a proper smooth curve of genus at least 2 contained in an abelian variety $A$ over a global field $K$ of characteristic $p > 0$. If there is a Zariski dense adelic point on $X$ which survives $p^n$-descent for all $n \geq 1$, then $X$ is isotrivial.*

The proof of this proposition will be given at the end of this section.

**Definition 2.4.** *Let $L \subset K$ be a subfield. We say that $X$ is* defined over *$L$ if there exists $X_0/L$ such that $X \simeq X_0 \times_K L$. We say that $X$ is* definable over *$L$ if there exists $X_0/L$ such that $X \times_K \overline{K} \simeq X_0 \times_L \overline{K}$, where $\overline{K}$ denotes an algebraic closure of $K$ containing $L$.*

For an abelian variety $A/K$, multiplication by $p^n$ factors as

$$A \xrightarrow{F^n} A^{(p^n)} \xrightarrow{V^n} A\,,$$

where $F^n$ and $V^n$ are the $n$-fold compositions of the absolute Frobenius and Verschiebung isogenies.

**Lemma 2.5.** *Suppose $X$ contains a Zariski dense adelic point which lifts to a $p^n$-covering $Y' \to X$ and let $Y \to X$ be the torsor under $\ker(V^n : A^{(p^n)} \to A)$ through which it factors. Then $Y_{\mathrm{red}}$ is geometrically reduced and definable over $K^{p^n}$.*

*Proof.* Let $(P_v)_v \in X(\mathbb{A}_K)$ be the given adelic point and let $Y' \to X$ be the $p^n$-covering to which $(P_v)_v$ lifts. By passing to a separable extension of $K$ (which is harmless because $(K^{\mathrm{sep}})^p \cap K = K^p$ and [Hug05, Lemma 1.5.11]) we can assume $Y' \to X$ is the pullback of multiplication by $p^n$ on $A$. In particular, it factors through the $n$-fold Frobenius morphism $F^n : A \to A^{(p^n)}$ and we have a commutative diagram with $Y$ the torsor in the statement,

$$\begin{array}{ccccc}
Y'_{\mathrm{red}} & \longrightarrow & Y_{\mathrm{red}} & \longrightarrow & X \\
\downarrow & & \downarrow & & \| \\
Y' & \longrightarrow & Y & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{F^n} & A^{(p^n)} & \xrightarrow{V^n} & A\,.
\end{array}$$

Let $(Q_v)_v \in Y'(\mathbb{A}_K)$ denote a lift of $(P_v)_v$. For any $v$, the point $Q_v : \mathrm{Spec}(K_v) \to Y'$ factors through $Y'_{\mathrm{red}}$, because $\mathrm{Spec}(K_v)$ is reduced. So $(Q_v)_v$ is also a Zariski dense adelic point on $Y'_{\mathrm{red}}$. Its image $(R_v)_v$ in $Y_{\mathrm{red}}(\mathbb{A}_K)$ is a Zariski dense adelic point and by commutativity of the diagram the image of $(R_v)_v$ in $A^{(p^n)}$ lies in $F^n(A(\mathbb{A}_K))$. In particular, for each $v$, the point $R_v$ lies in $A^{(p^n)}(K_v^{p^n})$. It then follows from the proof of [AV92, Lemma 1] that $Y_{\mathrm{red}}$ is defined over $K^{p^n}$ and is geometrically reduced. Below is an alternative argument using [Vol91], in particular, the last paragraph.

We show that that $Y_{\mathrm{red}}$ is defined over $K^{p^n}$ and is geometrically reduced. Assume $n = 1$, which is enough, as the argument can be repeated $n$ times. Let $U$ be an affine open subset of $Y_{\mathrm{red}}$ and $f$ a function defined on an affine open set of $A^{(p^n)}$ which vanishes on $U$. We have that $f(R_v) = 0$ and differentiating this equation with respect to a derivation $\delta$ on $K$ with kernel $K^p$, gives $f^\delta(R_v) = 0$. Since $(R_v)_v$ is Zariski dense on $Y_{\mathrm{red}}$, we conclude that $f^\delta$ also vanishes on $U$. This means that $\delta$ extends to a vector field on a spreading out of $Y_{\mathrm{red}}$ and we conclude via [Vol91, Lemma 1]. □

**Remark 2.6.** *From the above proof, if $Y_{\mathrm{red}}$ is not defined over $K^p$, some $f^\delta$ does not vanish on $Y_{\mathrm{red}}$ and the equation $f^\delta = 0$ defines a proper Zariski closed subset containing $(R_v)_v$.*

**Lemma 2.7.** *If $X' \to X$ is a torsor under an étale group scheme and $X'$ is definable over $K^{p^n}$, then $X$ is definable over $K^{p^n}$.*

*Proof.* [Vol91, Lemma 2] proves this for Galois covers. This gives the result, since taking a separable extension to trivialise the Galois action on the étale group scheme is harmless. □

**Lemma 2.8.** *Suppose $Y_i \subset A_i$ are geometrically integral curves contained in abelian varieties $A_i$ over $K$, for $i = 1, 2$. Suppose there is an isogeny $A_1 \to A_2$ restricting to a generically purely inseparable map $Y_1 \to Y_2$. If $Y_1$ and $A_1$ are definable over $K^{p^n}$, then $Y_2$ is definable over $K^{p^n}$.*

*Proof.* Passing to a finite separable extension we can assume $Y_1$ is defined over $K^{p^n}$. In particular, $Y_1$ is defined over $K^p$, so the argument in [AV92, Theorem A(2)] shows that $Y_2$ is defined over $K^p$. Replacing $K$ with $K^p$ and repeating $n$ times we find that $Y_2$ is defined over $K^{p^n}$. □

*Proof of Proposition 2.3.* Let $P := (P_v)_v \in X(\mathbb{A}_K)$ be a Zariski dense adelic point that survives $p^n$-descent for all $n \geq 1$.

Let $n \geq 1$ and let $Y' \to X$ be a $p^n$-covering to which $P$ lifts. By Lemma 2.5, $Y' \to X$ factors through a torsor $Y \to X$ under the kernel of $V^n : A^{(p^n)} \to A$, with $Y_{\mathrm{red}}$ geometrically reduced and definable over $K^{p^n}$. We can factor $V^n = V_e \circ V_c$, with $V_c$ an isogeny whose kernel is a connected abelian $p$-group scheme and $V_e$ étale. Let $Y \to X_e \to X$ be the corresponding factorization of $Y \to X$. Since $X_e \to X$ is étale and $X$ is smooth, $X_e$ is geometrically integral. The isogeny $V_c$ restricts to a morphism $Y_{\mathrm{red}} \to X_e$ which is generically purely inseparable, so $X_e$ is definable over $K^{p^n}$ by Lemma 2.8. Then $X$ is definable over $K^{p^n}$ by Lemma 2.7.

Since $P$ survives $p^n$-descent for all $n$, we conclude that $X$ is definable over $K^{p^n}$ for all $n \geq 1$. This implies that $X$ is isotrivial (see the discussion in [Szp81, Section 0]). □

## 3. Rational points on finite subschemes of abelian varieties

**Proposition 3.1.** *Let $Z \subset A$ be a finite subscheme of an abelian variety defined over a global function field $K$. Then*

$$Z(K) = Z(\mathbb{A}_K) \cap \overline{A(K)} = Z(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\mathrm{Br}}.$$

*Proof.* By [PV10, Theorem E] we have $Z(K) \subset \overline{A(K)} \subset A(\mathbb{A}_K)^{\mathrm{Br}} \subset \widehat{\mathrm{Sel}}(A)$. So it suffices to show that $Z(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(A) \subset Z(K)$. As in the proof of [PV10, Prop. 3.9], it suffices to show that this holds after a finite separable extension, so we can assume that $Z$ is a finite set of $K$-points.

Replacing $K$ by a further finite separable extension if needed, we can also assume that $A[n]$ is a constant group scheme for some $n$ prime to $p$ and that the Néron model of $A$ has semiabelian connected component. In the appendix by D. Rössler it is shown that, under these hypotheses, there exists an étale isogeny $f : A \to B$ and an isogeny $g : B \to B$ of degree $> 1$ such that $\ker(g)(K^{\mathrm{sep}}) = 0$. Let $W = f(Z) \subset B$. If $B(K^{\mathrm{sep}})[p] = 0$, then [PV10, Proposition 5.3] gives that $W(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(B) = W(K)$. Working with the given endomorphism $g$ instead of multiplication by $p$, the argument there can be adapted to give the same conclusion (Details are given in Lemma 3.2 below).

Now suppose $P \in Z(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(A)$. It follows from the definition of the Selmer groups that $f\left(\widehat{\mathrm{Sel}}(A)\right) \subset \widehat{\mathrm{Sel}}(B)$. So $f(P) \in W(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(B) = W(K)$. For any $v \in \Omega_K$, the $v$-adic component of $P$ is the image of some $Q_v \in Z(K)$. The adelic point $P - Q_v \in A(\mathbb{A}_K)$ lies in the kernel of $f$ and in $\widehat{\mathrm{Sel}}(A)$. So $P - Q_v \in \widehat{\mathrm{Sel}}(A)_{\mathrm{tors}}$. By [PV10, Lemma 5.1] this implies that $P - Q_v \in A(K)$. So $P \in A(K)$. ∎

Here are details of the claimed analogue of [PV10, Proposition 5.3] used in the proof above.

**Lemma 3.2.** *Let $W \subset B$ be a finite subscheme of an abelian variety defined over $K$. Suppose there exists an endomorphism $g : B \to B$ of degree $> 1$ such that $B(K^{\mathrm{sep}})[g] = 0$. Then $W(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(B) = W(K)$.*

*Proof.* We have

$$W(K) \subset B(K) \subset B(\mathbb{A}_K)^{\mathrm{Br}} \subset \widehat{\mathrm{Sel}}(B).$$

So it suffices to show that $W(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(B) \subset W(K)$. Moreover we can assume $W = W(K)$ as in [PV10, Prop. 3.9].

Suppose $P \in W(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(B)$. For any $v \in \Omega_K$, the $v$-adic component of $P$ is the image of some point $Q_v \in W(K)$, and $P - Q_v \in \widehat{\mathrm{Sel}}(B)$ maps to 0 in $B(K_v)^{(g)} := \varprojlim_n B(K_v)/g^n(B(K_v))$. In particular, $P - Q_v$ is in the kernel of $\mathrm{Sel}^{(g)}(B) \to B(K_v)^{(g)}$ where $\mathrm{Sel}^{(g)}(B)$ denotes the inverse limit of the Selmer groups corresponding to the isogenies $g^n$ for $n \geq 1$. Below we show that this map is injective, so the image of $P - Q_v$ in $\mathrm{Sel}^{(g)}(B)$ is 0.

Since this holds for any $v$, if $v'$ is any other prime we have

$$Q_{v'} - Q_v \in \ker\left(B(K) \to \varprojlim_n B(K)/g^n B(K) \hookrightarrow \mathrm{Sel}^{(g)}(B)\right).$$

In other words, $(Q_v - Q_{v'}) \in \cap_{n\geq 1} g^n B(K)$. Since $B(K)$ is finitely generated, this implies that $(Q_v - Q_{v'}) \in B(K)_{\mathrm{tors}}$. Again, since this holds for all $v$ we see that $R := P - Q_v \in \widehat{\mathrm{Sel}}(B)_{\mathrm{tors}}$. By [PV10, Lemma 5.1] this implies that $P - Q_v \in B(K)$. So $P \in W(K)$.

It remains to prove that $\mathrm{Sel}^{(g)}(B) \to B(K_v)^{(g)}$ is injective. For this it suffices (as in [PV10, Proof of 5.2]) to prove injectivity of $\mathrm{Sel}'^{(g)}(B) \to B(K'_v)^{(g)}$, where $K'_v \subset K^{\mathrm{sep}}$ denotes the Henselization with respect to $v$ and $\mathrm{Sel}'^{(g)}(B)$ is defined in the same way as $\mathrm{Sel}^{(g)}(B)$ but using $K'_v$ instead of $K_v$. Let $b \in \ker\left(\mathrm{Sel}'^{(g)}(B) \to B(K'_v)^{(g)}\right)$ and let $b_M$ denote its image in $\mathrm{Sel}'^{g^M}(B) \subset \mathrm{H}^1(K'_v, B[g^M])$. Then the image of $b_M$ under

$$\mathrm{Sel}'^{g^M}(B) \to \frac{B(K'_v)}{g^M B(K'_v)} \subset \mathrm{H}^1(K'_v, B[g^M]) \to \mathrm{H}^1(K^{\mathrm{sep}}, B[g^M])$$

is 0. The inflation-restriction sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), B(K^{\mathrm{sep}})[g^M]) \to \mathrm{H}^1(K, B[g^M]) \to \mathrm{H}^1(K^{\mathrm{sep}}, B[g^M])$$

shows that $b_M$ comes from an element of $\mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), B(K^{\mathrm{sep}})[g^M])$. But this group is trivial since $B(K^{\mathrm{sep}})[g^M] = 0$. Since this holds for all $M \geq 1$, we conclude that $b = 0$. ☐

## 4. Proofs of the theorems

*Proof of Theorem 1.2.* By [PV10, Theorem E] we have $\overline{A(K)} \subset A(\mathbb{A}_K)^{\mathrm{Br}} \subset \widehat{\mathrm{Sel}}(A) \subset A(\mathbb{A}_K)$. intersecting with $X(\mathbb{A}_K)$ we have

$$X(\mathbb{A}_K) \cap \overline{A(K)} \subset X(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\mathrm{Br}} \subset X(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(A)\,,$$

where the rightmost set consists of the adelic points on $X$ which survive $N$-descent for all $N \geq 1$ (relative to the embedding $X \subset A$). In particular, any $P \in X(\mathbb{A}_K) \cap \overline{A(K)}$ survives $p^n$-descent for all $n \geq 1$ . Since $X$ is not isotrivial, Proposition 2.3 implies that there is a finite subscheme $Z \subset X \subset A$ which contains $P$. Then $P \in Z(\mathbb{A}_K) \cap \overline{A(K)} = Z(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\mathrm{Br}} = X(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(A) = Z(K)$, where the final equality is Proposition 3.1. ☐

**Remark 4.1.** *The preceding proof shows that for $X \subset A$ as in Theorem 1.2 we have*

$$X(K) = X(\mathbb{A}_K) \cap \overline{A(K)} = X(\mathbb{A}_K) \cap A(\mathbb{A}_K)^{\mathrm{Br}} = X(\mathbb{A}_K) \cap \widehat{\mathrm{Sel}}(A)\,.$$

*Proof of Theorem 1.1.* Let $X/K$ be as in the statement and let $J = \mathrm{Jac}(X)$ be its Jacobian. It suffices to show that $X(\mathbb{A}_K)^{\mathrm{Br}} \subset X(K)$. Passing to some finite separable extension $L/K$ we can embed $X_L$ in $J_L$ via the Abel-Jacobi map corresponding to an $L$-rational point. If $P \in X(\mathbb{A}_K)^{\mathrm{Br}}$, then its image under the inclusion $X(\mathbb{A}_K) \subset X(\mathbb{A}_L) = X_L(\mathbb{A}_L)$ is orthogonal to $\mathrm{Br}(X_L)$ by [CV, Lemma 3.1]. By functoriality of the Brauer pairing and Remark 4.1 we have $X_L(\mathbb{A}_L)^{\mathrm{Br}} \subset X_L(\mathbb{A}_L) \cap J_L(\mathbb{A}_L)^{\mathrm{Br}} = X_L(L)$. Then $P$ is in $X(\mathbb{A}_K) \cap X(L)$ which is equal to $X(K)$ by [PV10, Lemma 3.2]. ☐

**Remark 4.2.** *In the proof of Theorem 1.1 just given* [CV, Lemma 3.1] *and* [PV10, Lemma 3.2] *allow us to pass to an extension over which $X$ can be embedded in its Jacobian. Alternatively one can use the following construction suggested to one of us by Poonen. Restriction of scalars gives a map $\mathrm{Res}_{L/K}(X_L) \to \mathrm{Res}_{L/K}(J_L)$. Composing this with the canonical map $X \to \mathrm{Res}_{L/K}(X_L)$ gives a closed immersion $X \to A$ into the abelian variety $A := \mathrm{Res}_{L/K}(J_L)$ over $K$. To prove Theorem 1.1 one can then apply Remark 4.1 to $X \subset A$.*

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## Appendix A. On abelian varieties with an infinite group of separable $p^\infty$-torsion points (by Damian Rössler)

If $n \in \mathbb{N}$, we write $[n]$ for the multiplication by $n$ endomorphism on an abelian variety. If $h$ is a finite endomorphism of an abelian variety $A$ over a field $L$, we write

$$A(L)[h^\ell] := \{x \in A(L) \,|\, h^{\circ \ell}(x) = 0\}$$

and

$$A(L)[h^\infty] := \{x \in A(L) \,|\, \exists n \in \mathbb{N} : h^{\circ n}(x) = 0\}.$$

Here $h^{\circ n}(x) := h(h(h(\cdots(x)\cdots)))$, where there are $n$ pairs of brackets. The notation $A(L)[n^\ell]$ (resp. $A(L)[n^\infty]$) will be a shorthand for $A(L)[[n]^\ell]$ (resp. $A(L)[[n]^\infty]$).

Let now $K_0$ be the function field of a smooth and proper curve $U$ over a finite field $\mathbb{F}$ of characteristic $p > 0$. Let $B$ be an abelian variety over $K_0$. Suppose that for some $n > 3$ prime to $p$, the group scheme $B[n]$ is constant and that the Néron model of $B$ over $U$ has a semiabelian connected component.

**Proposition A.1.** *There exists*

- *an abelian variety $C$ over $K_0$;*
- *an étale $K_0$-isogeny $\phi : B \to C$;*
- *an étale $K_0$-isogeny $f : C \to C$;*
- *a $K_0$-isogeny $g : C \to C$;*
- *a natural number $r \geq 0$*

*such that*

(a) $g \circ f = [p^r]$ *and* $g \circ f = f \circ g$;
(b) $C(K_0^{\mathrm{sep}})[p^\infty] = C(K_0^{\mathrm{sep}})[f^\infty] = C(\bar{K}_0)[f^\infty]$;
(c) $C(K_0^{\mathrm{sep}})[g^\infty] = 0$.

*Proof.* For $\ell \geq 0$, define inductively

$$B_0 := B$$

and

$$B_{\ell+1} := B_\ell/(B_\ell(K_0^{\mathrm{sep}})[p]).$$

For $\ell_2 \geq \ell_1$, let $\phi_{\ell_1,\ell_2} : B_{\ell_1} \to B_{\ell_2}$ be the (étale!) morphism obtained by composing the natural morphisms $B_{\ell_1} \to B_{\ell_1+1} \to \cdots \to B_{\ell_2}$. We first claim that

$$(\ker \phi_{\ell_1,\ell_2})(K_0^{\mathrm{sep}}) = B_{\ell_1}(K_0^{\mathrm{sep}})[p^{\ell_2-\ell_1}] \tag{A.1}$$

We prove the claim by induction on $\ell_2 - \ell_1$. For $\ell_2 - \ell_1 \leq 1$, the claim is true by definition. Suppose that $\ell_2 - \ell_1 \geq 1$. Let $x \in B(K_0^{\mathrm{sep}})[p^{\ell_2-\ell_1}]$. Then $[p^{\ell_2-\ell_1-1}](x) \in B(K_0^{\mathrm{sep}})[p]$ and thus

$$\phi_{\ell_1,\ell_1+1}([p^{\ell_2-\ell_1-1}](x)) = [p^{\ell_2-\ell_1-1}](\phi_{\ell_1,\ell_1+1}(x)) = 0.$$

Applying the inductive assumption to $\phi_{\ell_1,\ell_1+1}(x)$, we see that $\phi_{\ell_1+1,\ell_2}(\phi_{\ell_1,\ell_1+1}(x)) = \phi_{\ell_1,\ell_2}(x) = 0$. This proves that $(\ker\,\phi_{\ell_1,\ell_2})(K_0^{\rm sep}) \supseteq B_{\ell_1}(K_0^{\rm sep})[p^{\ell_2-\ell_1}]$. To prove the opposite inclusion, let $x \in (\ker\,\phi_{\ell_1,\ell_2})(K_0^{\rm sep})$. We compute

$$\phi_{\ell_1,\ell_2}(x) = \phi_{\ell_1+1,\ell_2}(\phi_{\ell_1,\ell_1+1}(x)) = 0,$$

which implies (by the inductive hypothesis) that

$$[p^{\ell_2-\ell_1-1}](\phi_{\ell_1,\ell_1+1}(x)) = \phi_{\ell_1,\ell_1+1}([p^{\ell_2-\ell_1-1}](x)) = 0,$$

which in turn implies that $[p]([p^{\ell_2-\ell_1-1}](x)) = [p^{\ell_2-\ell_1}](x) = 0$. This proves that $(\ker\,\phi_{\ell_1,\ell_2})(K_0^{\rm sep}) \subseteq B_{\ell_1}(K_0^{\rm sep})[p^{\ell_2-\ell_1}]$ and completes the proof of the claim.

Now we know that by the reasoning made in the last page of [Rös13], that there are only finitely many isomorphism classes of abelian varieties over $K_0$ in the sequence $\{B_\ell\}_{\ell\in\mathbb{N}}$. Let $C$ be an abelian variety over $K_0$, which appears at least twice in the sequence $\{B_\ell\}_{\ell\in\mathbb{N}}$. Let $n_2 > n_1$ be such that $C \simeq B_{n_1} \simeq B_{n_2}$. Then by construction (under the identification $C = B_{n_1}$)

$$\phi^{\circ\ell}_{n_1,n_2} = \phi_{n_1,n_1+\ell\cdot(n_2-n_1)} \tag{A.2}$$

for any $\ell \geq 1$ and thus

$$C(K_0^{\rm sep})[p^\infty] = C(K_0^{\rm sep})[\phi^\infty_{n_1,n_2}] \tag{A.3}$$

Now define $f := \phi_{n_1,n_2}$ and $r := n_2 - n_1$. Define $g$ as the only $K_0$-isogeny such that $g \circ f = [p^r]$.

Notice then that the identity $g \circ f = [p^r]$ implies the identity $f \circ g = [p^r]$. To see this last fact directly, recall first that there are natural injection of rings

$$\mathrm{End}_{K_0}(C) \hookrightarrow \mathrm{End}_{\bar{K}_0}(C_{\bar{K}_0}) \hookrightarrow \mathrm{End}_{\mathbb{Z}_t}(T_t(C(\bar{K}_0))) \hookrightarrow \mathrm{End}_{\mathbb{Q}_t}(T_t(C(\bar{K}_0)) \otimes \mathbb{Q}_t)$$

where $T_t(C(\bar{K}_0))$ is the classical Tate module of $C_{\bar{K}_0}$ and $t > 0$ is some prime number $\neq p$. Now if $M$ and $N$ are two square matrices of the same size with coefficients in a field of characteristic 0, such that $M \cdot N = p^r$, then $p^{-r}N$ is the inverse matrix of $M$ and thus $N \cdot M = p^r$. This fact combined with the existence of the above injections implies that $f \circ g = [p^r]$ if $g \circ f = [p^r]$.

We have already proven (a). Point (b) is contained in equation (A.3).

We now prove (c). Suppose that for some $\ell \geq 0$ and some $x \in C(K_0^{\rm sep})$, we have $g^{\circ\ell}(x) = 0$. Let $y \in (f^{\circ\ell})^{-1}(x) \subseteq C(K_0^{\rm sep})$. Then $g^{\circ\ell}(f^{\circ\ell}(y)) = [p^{r\ell}](y) = 0$. Hence $f^{\circ\ell}(y) = 0 = x$ by (A.1) and (A.2). $\square$

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