

THE BRAUER-MANIN OBSTRUCTION FOR CONSTANT CURVES OVER GLOBAL FUNCTION FIELDS

BRENDAN CREUTZ AND JOSÉ FELIPE VOLOCH

ABSTRACT. Let \mathbb{F} be a finite field and C, D smooth proper curves over \mathbb{F} and set $K = \mathbb{F}(D)$. We consider Brauer-Manin and abelian descent obstructions to the existence of rational points and to weak approximation for the curve $C \otimes_{\mathbb{F}} K$. In particular, we show that Brauer-Manin is the only obstruction to weak approximation and the Hasse principle in the case that the genus of D is less than that of C . We also show that we can identify the points corresponding to non-constant maps $D \rightarrow C$ using Frobenius descents.

1. INTRODUCTION

Let C be a smooth, irreducible, projective curve of genus > 1 over a global field K . The question of whether the Brauer-Manin obstruction is the only obstruction to the Hasse principle or weak approximation for C was raised around 1999 by Scharaschkin and Skorobogatov. Not much progress has been made in the number field case, but a substantial amount of numerical evidence has been obtained, notably [BS08]. By contrast, the question in the function field case has been settled affirmatively for “most” curves [PV10], namely those curves whose Jacobian does not have an isotrivial factor and satisfy a certain condition on the p -power torsion points, where p is the characteristic of K . The latter condition has largely been removed due to recent work of Rössler [Rösa]. However, these results do not address the case of isotrivial or even constant curves and the purpose of this paper is to consider this case.

To put our results in context we begin with a summary of the general situation, assuming that C is embedded in its Jacobian J and that J has finite Tate-Shafarevich group. If S_C is the set of primes of good reduction for (some model of) C , then there is a commutative diagram

$$\begin{array}{ccc} C(K) & \cdots\cdots\cdots\longrightarrow & J(K) \\ \vdots \downarrow & & \downarrow \\ \prod_{v \in S_C} C(\mathbb{F}_v) & \longrightarrow & \prod_{v \in S_C} J(\mathbb{F}_v), \end{array}$$

where \mathbb{F}_v denotes the residue field at the prime v of K . Scharaschkin [Sch99] considered the intersection $C^{\text{MW-Sieve}}$ of the topological closures of the images of the solid arrows in the product of the discrete topologies on $J(\mathbb{F}_v)$. Scharaschkin showed that if $C^{\text{MW-Sieve}}$ is empty, then the set $C(\mathbb{A}_K)^{\text{Br}}$ of adelic points orthogonal to the Brauer group of C is too, and used this to give examples of curves of genus at least 2 which are counterexamples to the Hasse principle explained by the Brauer-Manin obstruction. Poonen conjectured, in the number field case, that every counterexample to the Hasse principle could be explained this way [Poo06]. Around the same time Stoll conjectured, in the number field case, that the set $C(\mathbb{A}_K)^{\text{Br}}$ of adelic points, modified at archimedean primes, orthogonal to the Brauer group

of C is equal to the (topological closure of the) image of $C(K)$ [Sto07]. Recall that $C(K)$ can be infinite when C is an isotrivial curve over a global function field. For an example showing that it is in general necessary to take the topological closure see [PV10, Remark 1.3]. One may also ask if $C^{\text{MW-Sieve}}$ is equal to the topological closure of the image of $C(K)$ in $\prod_{v \in S_C} J(\mathbb{F}_v)$. There are logical dependencies between these statements as follows.

$$(1.1) \quad \boxed{\overline{C(K)} = C(\mathbb{A}_K)^{\text{Br}} \text{ in } C(\mathbb{A}_K)} \implies \boxed{C(K) = \emptyset \Rightarrow C(\mathbb{A}_K)^{\text{Br}} = \emptyset}$$

$$\boxed{\overline{C(K)} = C^{\text{MW-Sieve}} \text{ in } \prod_{v \in S_C} C(\mathbb{F}_v)} \implies \boxed{C(K) = \emptyset \Rightarrow C^{\text{MW-Sieve}} = \emptyset}$$

$$\begin{array}{c} \uparrow \\ \uparrow \end{array}$$

All four statements are known to hold when $J(K)$ is finite by a result of Scharaschkin [Sch99]. The statement in the upper-left is the conclusion of the theorem of Poonen-Voloch mentioned above; the theorem says nothing about the statement in the lower-right.

1.1. Results for constant curves over global function fields. Let \mathbb{F} be a finite field and let C, D be smooth proper curves over \mathbb{F} . Set $K = \mathbb{F}(D)$. We consider Brauer-Manin and finite descent obstructions to the existence of K -rational points on the curve $C \otimes_{\mathbb{F}} K$, which we also denote by C . We remark that C may be embedded in its Jacobian J since it has a 0-cycle of degree 1 defined over \mathbb{F} by the Hasse-Weil bounds and that the Tate-Shafarevich group of J is finite by results of Tate and Milne [Mil68, Theorem 3].

Let $C(\mathbb{A}_{K, \mathbb{F}}) := \prod_v C(\mathbb{F}_v) \subset C(\mathbb{A}_K)$ be the set of reduced adelic points on C (see Section 2.2 for details). Since C can be defined over \mathbb{F} we have a model with everywhere good reduction and so $C^{\text{MW-Sieve}} \subset C(\mathbb{A}_{K, \mathbb{F}}) \subset C(\mathbb{A}_K)$.

Theorem 1.1. $C(\mathbb{A}_K)^{\text{Br}} = C(K) \cup C^{\text{MW-Sieve}}$.

In Corollary 4.3 we will deduce from Theorem 1.1 that the statements on the left in (1.1) are equivalent in the case of constant curves over global function fields (and similarly for those on the right, though this was already known [CVV18, Proposition 2.2]).

Our proof of Theorem 1.1 utilizes the connection between the Brauer-Manin and finite abelian descent obstructions developed in [Sto07] building on work of Colliot-Thélène and Sansuc, Harari and Skorobogatov. In particular, $C(\mathbb{A}_K)^{\text{Br}}$ is the set of adelic points surviving all torsors $C' \rightarrow C$ arising as pullbacks of isogenies $J \rightarrow J$ (See Proposition 2.2). So Theorem 1.1 is a consequence of the following two theorems proven in Sections 3 and 4, respectively.

Theorem 1.2. Let $C(\mathbb{A}_K)^{F^\infty}$ denote the set of adelic points surviving the n -th iterate of the \mathbb{F} -Frobenius isogeny $F : J \rightarrow J$ for all $n \geq 1$. Then

$$C(\mathbb{A}_K)^{F^\infty} = C(K) \cup C(\mathbb{A}_{K, \mathbb{F}}).$$

Theorem 1.3. Let $C(\mathbb{A}_K)^{\text{ét-isog}}$ denote the set of adelic points surviving all torsors arising as pullbacks of étale isogenies $J' \rightarrow J$. Then

$$C(\mathbb{A}_K)^{\text{ét-isog}} \cap C(\mathbb{A}_{K, \mathbb{F}}) = C^{\text{MW-Sieve}}.$$

This result remains true if $C(\mathbb{A}_K)^{\text{ét-isog}}$ is replaced by the a priori smaller set of adelic points surviving all torsors under abelian group schemes over K (see Theorem 4.2). Propositions 4.4

and 4.5 show that in general non-étale torsors are required to cut out the set of rational points (even if one includes non-abelian group schemes).

In Corollary 5.3 we give a characterization of the elements of $C^{\text{MW-Sieve}}$ in terms of maps between the sets $D(\overline{\mathbb{F}}) \rightarrow C(\overline{\mathbb{F}})$ which induce homomorphisms of their Jacobians. We are thus lead to ask if all such maps arise from a global point.

Question 1.4. *Suppose $\psi : D(\overline{\mathbb{F}}) \rightarrow C(\overline{\mathbb{F}})$ is a Galois-equivariant map of sets which, when extended linearly to divisors, sends principal divisors to principal divisors. Is there a morphism of curves $\phi : D \rightarrow C$ such that ψ is given by composing ϕ with a limit of Frobenius maps?*

An affirmative answer to this question implies that all four statements in (1.1) hold in the case of constant curves over global function fields. In Section 5 we show that the answer to this question is affirmative when the genus of D is less than the genus of C .

Theorem 1.5. *If the genus of D is less than the genus of C , then $C(\mathbb{A}_K)^{\text{Br}} = C(K) = C(\mathbb{F})$.*

2. NOTATION AND PRELIMINARIES

As in the introduction C will denote a constant curve over the global function field $K = \mathbb{F}(D)$ and J is the Jacobian of C . The places of K are in bijection with the set D^1 of closed points of D . Given a closed point $v \in D^1$ we use K_v , \mathcal{O}_v and \mathbb{F}_v to denote the corresponding completion, ring of integers and residue field, respectively. Let us fix once and for all an algebraic closure $\overline{\mathbb{F}}$ of \mathbb{F} and for each place v , an embedding $\mathbb{F} \subset \mathbb{F}_v \subset \overline{\mathbb{F}}$. The embedding determines a geometric point $\bar{v} \in D(\overline{\mathbb{F}})$ in the support of v . Let $G_{\mathbb{F}} = \text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ be the absolute Galois group of \mathbb{F} .

Throughout the paper X denotes a proper geometrically integral variety over \mathbb{F} and $X_K := X \otimes_{\mathbb{F}} K$.

2.1. Adelic points. The adèle ring of K is the K -algebra $\mathbb{A}_K = \prod_{v \in D^1} (K_v : \mathcal{O}_v)$, where the restricted product runs over the closed points of D . For any place v of K , the inclusions $\mathbb{F} \subset \mathbb{F}_v \subset \mathcal{O}_v \subset K_v$ endow \mathcal{O}_v, K_v and \mathbb{A}_K with the structure of \mathbb{F} -algebra. Consequently the sets

$$\begin{aligned} X(K_v) &:= \text{Mor}_{\text{Spec}(\mathbb{F})}(\text{Spec}(K_v), X), \\ X(\mathbb{A}_K) &:= \text{Mor}_{\text{Spec}(\mathbb{F})}(\text{Spec}(\mathbb{A}_K), X) \end{aligned}$$

are well defined. The universal property of fibered products gives canonical bijections of these sets with $X_K(K_v)$ and $X_K(\mathbb{A}_K)$, respectively. Since X is proper we may identify $X(\mathbb{A}_K) = X_K(\mathbb{A}_K) = \prod_{v \in D^1} X(K_v) = \prod_{v \in D^1} X(\mathcal{O}_v)$.

2.2. Reduced adelic points. The reduced adèle ring of K is the \mathbb{F} -algebra $\mathbb{A}_{K, \mathbb{F}} = \prod_{v \in D^1} \mathbb{F}_v$. This is an \mathbb{F} -subalgebra of \mathbb{A}_K . The set $X(\mathbb{A}_{K, \mathbb{F}}) = \text{Mor}_{\text{Spec}(\mathbb{F})}(\text{Spec}(\mathbb{A}_{K, \mathbb{F}}), X)$ of reduced adelic points on X is a closed subset of $X(\mathbb{A}_K)$ which can be identified with $\prod_{v \in D^1} X(\mathbb{F}_v)$, where the latter is endowed with the product of the discrete topologies. This agrees with the subspace topology determined by the inclusion $X(\mathbb{A}_{K, \mathbb{F}}) \subset X(\mathbb{A}_K)$. The quotient of \mathcal{O}_v by its maximal ideal induces the reduction map $r_v : X(K_v) = X(\mathcal{O}_v) \rightarrow X(\mathbb{F}_v)$. These give rise to a continuous projection $r : X(\mathbb{A}_K) \rightarrow X(\mathbb{A}_{K, \mathbb{F}})$ sending (x_v) to $(r_v(x_v))$.

Lemma 2.1. *A reduced adelic point $(x_v) \in X(\mathbb{A}_{K,\mathbb{F}})$ determines a unique map of sets $\psi : D(\overline{\mathbb{F}}) \rightarrow X(\overline{\mathbb{F}})$ with the property that $\psi(\bar{v}) = x_v$. This induces a bijection $X(\mathbb{A}_{K,\mathbb{F}}) \leftrightarrow \text{Map}_{G_{\mathbb{F}}}(D(\overline{\mathbb{F}}), X(\overline{\mathbb{F}}))$.*

Proof. Since $D(\overline{\mathbb{F}})$ is the union, as v ranges over the closed points of D , of the $G_{\mathbb{F}}$ -orbits of the points \bar{v} , it is clear that there is a unique $G_{\mathbb{F}}$ -equivariant map with the stated property. Conversely, given a map of $G_{\mathbb{F}}$ -sets $\psi : D(\overline{\mathbb{F}}) \rightarrow X(\overline{\mathbb{F}})$, we define an adelic point $(x_v) \in X(\mathbb{A}_{K,\mathbb{F}})$ by $x_v = \psi(\bar{v}) \in X(\overline{\mathbb{F}})$. Galois equivariance of the map shows that $x_v \in X(\mathbb{F}_v)$. \square

2.3. Rational points. The universal property of fibered products and the valuative criterion for properness give identifications $X_K(K) = X(K) = \text{Mor}_{\mathbb{F}}(\text{Spec}(K), X) = \text{Mor}_{\mathbb{F}}(D, X)$. Together with previous lemma we have a commutative diagram

$$\begin{array}{ccc} X(K) & \xrightarrow{r} & X(\mathbb{A}_{K,\mathbb{F}}) \\ \parallel & & \parallel \\ \text{Mor}_{\mathbb{F}}(D, X) & \longrightarrow & \text{Map}_{G_{\mathbb{F}}}(D(\overline{\mathbb{F}}), X(\overline{\mathbb{F}})) \end{array}$$

where the bottom map is the obvious one taking a morphism of varieties to the map it induces on geometric points. Since a morphism of varieties over a field is determined by what it does to geometric points, the horizontal maps are injective.

2.4. Brauer-Manin and abelian descent obstructions. Consider the category $\text{Cov}(X_K)$ of X_K -torsors under finite group schemes over K (see [Sto07, Section 4]). We say that an adelic point $P \in X(\mathbb{A}_K) = X_K(\mathbb{A}_K)$ survives $(X', G) \in \text{Cov}(X_K)$ if the element of $\prod_v H^1(K_v, G)$ given by evaluating (X', G) at P lies in the image of the diagonal map $H^1(K, G)$. Equivalently P survives (X', G) if and only if P lifts to an adelic point on some twist of (X', G) . The set of adelic points surviving a set of torsors is a closed subset of $X(\mathbb{A}_K)$ containing K . Let $X(\mathbb{A}_K)^{\text{ab}}$ denote the set of adelic points surviving all $(X', G) \in \text{Cov}(X_K)$ for which G is a finite abelian group scheme over K .

When X is a subvariety of an abelian variety A/\mathbb{F} we shall consider the subset of torsors in $\text{Cov}(X_K)$ which arise as pullbacks of (étale) isogenies $\phi : A' \rightarrow A$ defined over \mathbb{F} . We note that these are geometrically connected torsors under finite abelian group schemes over \mathbb{F} . They depend on the embedding $X \rightarrow A$, but only up to twist by elements of $H^1(\mathbb{F}, \ker(\phi))$. As such the sets $X(\mathbb{A}_K)^{\text{isog}}$ and $X(\mathbb{A}_K)^{\text{ét-isog}}$ of adelic points surviving all such torsors do not depend on the embedding.

Proposition 2.2. *Suppose X/\mathbb{F} is either a curve or an abelian variety. Then $X(\mathbb{A}_K)^{\text{Br}} = X(\mathbb{A}_K)^{\text{ab}} = X(\mathbb{A}_K)^{\text{isog}}$. If X is an abelian variety, then $X(\mathbb{A}_K)^{\text{Br}} = \overline{X(K)}$.*

Proof. Stoll proved the number field analogue of the first statement [Sto07, Section 7]. For the extension to global function fields see [CVV18, Section 2]. For the second statement see [PV10, Remark 4.5]. \square

3. FROBENIUS DESCENT OBSTRUCTION

Let J be the Jacobian of C and fix an embedding $C \rightarrow J$. Consider the \mathbb{F}_p -Frobenius isogeny $F : J^{(-1)} \rightarrow J$. Zariski locally, defining equations for J are obtained from those defining $J^{(-1)}$ by taking p -th powers and F is given by raising coordinates to their p -th

powers. The pullback of F is a torsor $(C', \ker(F)) \in \text{Cov}(C_K)$ under the finite abelian \mathbb{F} -group scheme $\ker(F) \subset J^{(-1)}$. The torsor $(C', \ker(F))$ does not depend on the choice of embedding because $F : J^{(-1)}(\mathbb{F}) \rightarrow J(\mathbb{F})$ is surjective. We note that C' is not reduced; the induced morphism $C'_{\text{red}} \rightarrow C$ on the reduced subscheme of C' is the \mathbb{F}_p -Frobenius morphism $C^{(-1)} \rightarrow C$ which has degree p , while $C' \rightarrow C$ has degree p^g .

Lemma 3.1. *There is a functorial (on separable extensions L/K) injection $\mu_L : H^1(L, \ker F) \rightarrow \Omega_{L/\mathbb{F}}^{\oplus g}$ such that the induced map $J(L)/F(J^{(-1)}(L)) \rightarrow \Omega_{L/\mathbb{F}}^{\oplus g}$ is given by $x \mapsto x^*(\omega_1, \dots, \omega_g)$ where $\omega_1, \dots, \omega_g$ is a basis of holomorphic differentials of C (so of J as well).*

The existence of the map is a slight restatement of [Rös, Proposition 1.1] (which is, in turn, a variant of [AM76, §2]) and the explicit expression follows from the proof there. See also [BV95, pg 123-124] for a proof in the ordinary case.

Lemma 3.2. *If $(x_v) \in C(\mathbb{A}_K)$ survives $(C', \ker(F))$ and $\mu_{K_v}(x_v) \neq 0$ for some v , then $(x_v) \in C(K)$ unless $p = 2$ and C is hyperelliptic, in which case a similar result holds with F replaced by F^2 .*

Proof. If $x = (x_v) \in C(\mathbb{A}_K)$ survives F -descent, then $(x_v) = y + F(z_v)$, for some $y \in J(K)$ and $(z_v) \in J^{(-1)}(\mathbb{A}_K)$. For any v we have that $\mu_{K_v}(x_v) = x_v^*(\omega_1, \dots, \omega_g) = y^*(\omega_1, \dots, \omega_g) = \mu_K(y)$. Choose a separating variable $t \in K$ and write $\omega_i = f_i dt$ with $f_i \in \mathbb{F}(C)$.

Consider the case that p is odd or C is not hyperelliptic. Suppose $\mu_K(y)$ is non-zero and consider the point $P = (f_1(x_v) : \dots : f_g(x_v)) = (f_1(y) : \dots : f_g(y)) \in \mathbb{P}^{g-1}$. From the right hand side we see that $P \in \mathbb{P}^{g-1}(K)$. From the left hand side P is the image of x_v under the canonical morphism $C \rightarrow \mathbb{P}^{g-1}$, and this immediately gives that (x_v) is a global point, unless C is hyperelliptic and $C \rightarrow \mathbb{P}^{g-1}$ is not ramified above P . In this case the fiber of $C \rightarrow \mathbb{P}^{g-1}$ above P is a locally trivial torsor under $\mathbb{Z}/2\mathbb{Z}$ and, hence, consists of a pair of global points $Q, Q' \in C(K)$ interchanged by the hyperelliptic involution. Since $Q + Q'$ is linearly equivalent to an \mathbb{F} -rational divisor it lies in $F(J^{(-1)}(\mathbb{F})) \subset \ker(\mu_K)$ and so $\mu(Q) = -\mu(Q')$. Since p is odd, this shows that $\mu(Q) \neq \mu(Q')$. Since $x_v \in \{Q, Q'\}$ and $\mu_{K_v}(x_v) = \mu_K(y)$ does not depend on v , we conclude that all x_v must be equal and so $(x_v) \in C(K)$. For hyperelliptic curves in characteristic 2 a similar argument using F^2 -descent taking values in a module over the ring $W_2(K)$ of length two Witt vectors gives the result (we omit details). \square

Remark 3.3. *In the case $p = 2$ and C is hyperelliptic the proof shows that an adelic point (x_v) surviving $(C', \ker(F))$ which does not lift to C' has $x_v \in C(K)$ for each v and in particular that $C(K) \neq \emptyset$. The issue is that the various x_v may differ from one another by the hyperelliptic involution and so (x_v) may not be global.*

Corollary 3.4. *The nontrivial twists of the Frobenius torsor $C' \rightarrow C$ satisfy the Hasse principle.*

Proof. If X is such a torsor and contains an adelic point, then this point maps to an adelic point of C unobstructed by F . Hence, C contains a global point that is the image of a K_v -point on X by the lemma (and the remark above). Since $X \rightarrow C$ is purely inseparable, this implies that X contains a global point as well. \square

Note that whether or not the trivial torsor satisfies the Hasse principle depends on whether or not C itself does.

Proof of Theorem 1.2. Let $(x_v) \in C(\mathbb{A}_K)^{F^\infty}$. In particular, (x_v) survives $(C', \ker(F))$. If $\mu_{K_v}(x_v) \neq 0$, then (x_v) is global by the lemma. Otherwise $(x_v) \in F(J^{(-1)}(\mathbb{A}_K))$ in which case (x_v) lifts to some $(y_v) \in C^{(-1)}(\mathbb{A}_K)$. By [GAT12, Main Theorem] $\text{III}^1(K, \ker(F)) = \ker(\text{H}^1(K, \ker(F)) \rightarrow \prod_v \text{H}^1(K_v, \ker(F))) = 0$, so (x_v) does not lift to any nontrivial twist of $(C', \ker(F))$. It follows that $(y_v) \in C^{(-1)}(\mathbb{A}_K)^{F^\infty}$. This argument may be iterated, so we conclude that (x_v) is either global or arbitrarily divisible by Frobenius, hence in $C(\mathbb{A}_{K,\mathbb{F}})$. This proves that $C(\mathbb{A}_K)^{F^\infty} \subset C(K) \cup C(\mathbb{A}_{K,\mathbb{F}})$. For the reverse inclusion note that $C'(\mathbb{A}_{K,\mathbb{F}}) = C^{(-1)}(\mathbb{A}_{K,\mathbb{F}})$ and the map $F : C'(\mathbb{A}_{K,\mathbb{F}}) \rightarrow C(\mathbb{A}_{K,\mathbb{F}})$ agrees with the \mathbb{F}_p -Frobenius map $C^{(-1)}(\mathbb{A}_{K,\mathbb{F}}) \rightarrow C(\mathbb{A}_{K,\mathbb{F}})$ which is surjective. Hence $C(\mathbb{A}_{K,\mathbb{F}}) \subset C(\mathbb{A}_K)^{F^\infty}$. \square

Remark 3.5. *It follows from the proof of Theorem 1.2 that the subset of $C(K)$ corresponding to non-constant maps $D \rightarrow C$ is characterized as the set of adelic points surviving F^∞ -descent that, at some stage, lift to a non-trivial torsor.*

4. ÉTALE ABELIAN DESCENT OBSTRUCTION

In this section we show that, at the level of reduced adelic points, all of the information given by the Brauer group can be obtained from finite abelian and étale torsors. We use $X(\mathbb{A}_{K,\mathbb{F}})^*$ to denote $X(\mathbb{A}_K)^* \cap X(\mathbb{A}_{K,\mathbb{F}})$.

Proposition 4.1. *Suppose X is a subvariety of an abelian variety. Then $X(\mathbb{A}_{K,\mathbb{F}})^{\text{isog}} = X(\mathbb{A}_{K,\mathbb{F}})^{\text{ét-isog}}$.*

Proof. Suppose $P \in X(\mathbb{A}_{K,\mathbb{F}})^{\text{ét-isog}}$ and let (X', G) be the pullback of some isogeny $\phi : A' \rightarrow A$. We will show that P survives (X', G) . Since G/\mathbb{F} is abelian it decomposes as a direct product $G = G_c \times G_e$ of a connected group scheme and an étale group scheme [Mil17, Proposition 11.3]. This gives rise to an étale torsor $(X'/G_e, G_e)$ and an inseparable torsor $(X'/G_e, G_c)$, and it suffices to show that P survives both. It survives the first by assumption. Evaluation of the second torsor at P gives an element of $\prod_{v \in D^1} \text{H}^1(\mathbb{F}_v, G_c)$. But $\text{H}^1(\mathbb{F}_v, G_c) = 0$ since G_c is connected and \mathbb{F}_v is perfect. So P lifts to $(X'/G_e, G_c)$ as well. \square

Theorem 4.2. *If X/\mathbb{F} is either a curve or an abelian variety, then $X(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}} = X(\mathbb{A}_{K,\mathbb{F}})^{\text{ab}} = X(\mathbb{A}_{K,\mathbb{F}})^{\text{ét-isog}}$. If X is a curve, then also $X(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}} = X^{\text{MW-Sieve}}$.*

Proof. The first statement follows from Propositions 2.2 and 4.1. The proof of [CVV18, Proposition 2.12] gives the second. \square

This proves Theorem 1.3 and, consequently, Theorem 1.1, which has the following Corollary.

Corollary 4.3. *Suppose C/\mathbb{F} is a curve. Then $\overline{C(K)} = C(\mathbb{A}_K)^{\text{Br}}$ in $C(\mathbb{A}_K)$ if and only if $C^{\text{MW-Sieve}} = \overline{r(C(K))}$ in $C(\mathbb{A}_{K,\mathbb{F}})$.*

Proof. Assume $\overline{C(K)} = C(\mathbb{A}_K)^{\text{Br}}$ and let $P \in C^{\text{MW-Sieve}}$. By Theorem 1.1 there exist $P_n \in C(K)$ such that $P_n \rightarrow P$ in $C(\mathbb{A}_K)$. Since $r : C(\mathbb{A}_K) \rightarrow C(\mathbb{A}_{K,\mathbb{F}})$ is continuous, $r(P_n) \rightarrow r(P) = P$, so $P \in \overline{r(C(K))}$.

Conversely, suppose $C^{\text{MW-Sieve}} = \overline{r(C(K))}$ and let $P \in C(\mathbb{A}_K)^{\text{Br}}$. By Theorem 1.1 we have $P \in C(K)$ or $P \in C^{\text{MW-Sieve}}$. The result holds trivially in the former case, so suppose $P \in C^{\text{MW-Sieve}}$. By assumption there are $P_n \in C(K)$ such that $r(P_n) \rightarrow P$ in the subspace

$C(\mathbb{A}_{K,\mathbb{F}}) \subset C(\mathbb{A}_K)$. By [CVV18, Lemma 2.13] the sequence $F^{n!} : C(\mathbb{A}_K) \rightarrow C(\mathbb{A}_K)$ of $n!$ -th iterates of the \mathbb{F} -Frobenius converges uniformly to r . Hence $\lim_n F^{n!} P_n = P$, showing that P lies in the closure of $C(K)$ in $C(\mathbb{A}_K)$. \square

Theorem 4.2 does not hold if one replaces $X(\mathbb{A}_{K,\mathbb{F}})$ with $X(\mathbb{A}_K)$.

Proposition 4.4. *Let X/\mathbb{F} be an abelian variety of p -rank 0 with $X(K) = X(\mathbb{F})$, then $X(K) = X(\mathbb{A}_K)^{\text{Br}} \neq X(\mathbb{A}_K)^{\text{ét-isog}}$.*

Proof. Let $(x_v) \in X(\mathbb{A}_K)$ be such that $r(x_v) = 0_X$. For each v , the kernel of reduction in $X(K_v)$ is divisible by all integers prime to p . So, for any n prime to p we have $(x_v) \in nX(\mathbb{A}_K)$. The assumption on the p -rank implies that all (geometrically connected) torsors under étale abelian group schemes are dominated by a twist of $[n] : X \rightarrow X$ for some n prime to p and so $(x_v) \in X(\mathbb{A}_K)^{\text{ét-isog}}$. Then $X(\mathbb{A}_K)^{\text{ét-isog}}$ is infinite, while $X(K) = X(\mathbb{F})$ is not. \square

In the case of curves the étale torsors do not always cut out the set of rational points, even if one allows torsors under finite non-abelian group schemes that are not required to be geometrically connected.

Proposition 4.5. *There exists a curve C/\mathbb{F} such that $C(K) = C(\mathbb{A}_K)^{\text{Br}} \neq C(\mathbb{A}_K)^{\text{ét}}$.*

Proof. Take $K = \mathbb{F}_p(t)$, $p \neq 2, 5$ and $X : y^2 = x^5 + 1$, so $C(K) = C(\mathbb{F})$ and $J(K) = J(\mathbb{F})$ are finite. The latter implies that $C(\mathbb{A}_K)^{\text{Br}} = C(K)$ by a result of Schraschkin. Now, for every place v of K choose a local parameter t_v and let $x_v = (t_v, \sum_j \binom{1/2}{j} t_v^{5j})$. Then the adelic point (x_v) survives all finite étale torsors, lifting to the same twist that $r((x_v)) = (0, 1) \in C(\mathbb{F}) \subset C(K)$ does. \square

5. REDUCED ADELIC POINTS AND MAPS

Let J_D be the Jacobian of D and fix an embedding $D \rightarrow J_D$ corresponding to a 0-cycle of degree 1 (which exists by the Hasse-Weil bounds). Recall that by Lemma 2.1 there is a bijection between the set $X(\mathbb{A}_{K,\mathbb{F}})$ of reduced adelic points and the set $\text{Map}_{G_{\mathbb{F}}}(D(\overline{\mathbb{F}}), X(\overline{\mathbb{F}}))$ of Galois equivariant maps on geometric points.

Theorem 5.1. *Let A/\mathbb{F} be an abelian variety. For any pair $(\psi, P) \in \text{Hom}_{G_{\mathbb{F}}}(J_D(\overline{\mathbb{F}}), A(\overline{\mathbb{F}})) \times A(\mathbb{F})$ the map $\psi|_D + P : D(\overline{\mathbb{F}}) \rightarrow A(\overline{\mathbb{F}})$ corresponds to a reduced adelic point which survives all torsors in $\text{Cov}(A_K)$. This induces bijections*

$$\overline{r(A(K))} = A(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}} \leftrightarrow \text{Hom}_{G_{\mathbb{F}}}(J_D(\overline{\mathbb{F}}), A(\overline{\mathbb{F}})) \times A(\mathbb{F}).$$

Proof. Let $(x_v) \in A(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}}$ and let $\psi : D(\overline{\mathbb{F}}) \rightarrow A(\overline{\mathbb{F}})$ be the corresponding Galois-equivariant map. Extending by linearity we obtain a Galois-equivariant homomorphism $\psi' : Z^0(\overline{D}) \rightarrow Z^0(\overline{A})$ on the groups of 0-cycles. The map sending a 0-cycle $z \in Z^0(\overline{A})$ to the pair $(\text{sum}(z), \text{deg}(z)) \in A(\overline{\mathbb{F}}) \times \mathbb{Z}$ induces an isomorphism of the group of 0-cycles on \overline{A} modulo albanese equivalence onto $A(\overline{\mathbb{F}}) \times \mathbb{Z}$.

We claim that if $z \in Z^0(\overline{D})$ is a principal divisor, then the 0-cycle $\psi'(z)$ is albanese equivalent to 0. To see this, suppose $z = \sum n_P P$ where P are geometric points of D . Let F^n be a power of the \mathbb{F} -Frobenius which fixes all P appearing in the support of z . By Theorem 4.2, $(x_v) \in A(\mathbb{A}_{K,\mathbb{F}})^{\text{ét-isog}}$ so it must lift to a twist of $(1 - F^n) : A \rightarrow A$. Since $\text{III}(K, A)$ is finite [Mil68, Theorem 3] we may choose n large enough so that the twist to which it lifts contains a rational point, and hence be of the form $(1 - F^n) + \gamma : A \rightarrow A$

for some $\gamma \in A(K)$. Evaluating at a point $P \in D(\overline{\mathbb{F}})$ in the support of z we conclude that $\psi(P) = (1 - F^n)(P) + \gamma(P) = \gamma(P)$. By linearity it follows that $\psi(z) = \gamma(z)$. By the universal property of the Jacobian of D , $\gamma \in A(K) = \text{Mor}_{\mathbb{F}}(D, A)$ factors through a morphism $\gamma_0 : J_D \rightarrow A$ of abelian varieties. Hence, if the image of z in J_D is trivial, its image in A must be trivial as well.

The claim established in the previous paragraph implies that ψ' factors through a Galois-equivariant map $\text{Pic}(\overline{D}) \rightarrow A(\overline{\mathbb{F}}) \times \mathbb{Z}$. We obtain a pair $(\psi, P) \in \text{Hom}_{G_{\overline{\mathbb{F}}}}(J_D(\overline{\mathbb{F}}), A(\overline{\mathbb{F}})) \times A(\mathbb{F})$ by taking ψ as the restriction of the above map to $\text{Pic}^0(\overline{D}) = J_D(\overline{\mathbb{F}})$ and defining P by $\psi'(z_1) = (P, 1) \in A(\overline{\mathbb{F}}) \times \mathbb{Z}$, where $z_1 \in Z^0(D)$ is the 0-cycle of degree 1 used to embed D in J_D . This yields the map $A(\mathbb{A}_{K, \mathbb{F}})^{\text{Br}} \rightarrow \text{Hom}_{G_{\overline{\mathbb{F}}}}(J_D(\overline{\mathbb{F}}), A(\overline{\mathbb{F}})) \times A(\mathbb{F})$.

Now suppose $(\psi, P) \in \text{Hom}_{G_{\overline{\mathbb{F}}}}(J_D(\overline{\mathbb{F}}), A(\overline{\mathbb{F}})) \times A(\mathbb{F})$. The map $\psi|_D + P : D(\overline{\mathbb{F}}) \rightarrow A(\overline{\mathbb{F}})$ corresponds to a reduced adelic point of A and it is easy to check that this yields an inverse to the map constructed above. We will show that the adelic point corresponding to $\psi|_D + P$ lies in the closure of the image of $A(K)$ in $A(\mathbb{A}_{K, \mathbb{F}})$. This suffices to prove the theorem since $A(\mathbb{A}_{K, \mathbb{F}})^{\text{Br}}$ is a closed set containing the image of $A(K)$. Furthermore, we may assume $P = 0$.

The homomorphism $\psi : J_D(\overline{\mathbb{F}}) \rightarrow A(\overline{\mathbb{F}})$ induces also a morphism $\psi' \in \text{Hom}_{G_{\overline{\mathbb{F}}}}(T_{\text{ét}}J_D, T_{\text{ét}}A)$ between the full étale Tate modules of J_D and A , i.e., $T_{\text{ét}}A = \varprojlim_n A(\overline{\mathbb{F}})[n]$. Since \mathbb{F} is perfect, the abelian group schemes $A[n]$ split as a direct product of an étale and a connected group scheme [Mil17, Proposition 11.3]. It follows that the full Tate module (profinite group scheme) splits as $TA = T_{\text{ét}}A \times A_0$, with A_0 a connected pro- p group scheme and similarly for J_D . As there are no nontrivial morphisms between étale and connected group schemes we obtain a surjective map $\text{Hom}_{\mathbb{F}}(TJ_D, TA) \rightarrow \text{Hom}_{G_{\overline{\mathbb{F}}}}(T_{\text{ét}}J_D, T_{\text{ét}}A)$. Tate's isogeny theorem [Tat66, WM71] gives an isomorphism $\text{Hom}_{\mathbb{F}}(J_D, A) \otimes \hat{\mathbb{Z}} \rightarrow \text{Hom}_{G_{\overline{\mathbb{F}}}}(TJ_D, TA)$. From this it follows that, for every n , there is some $\phi_n \in \text{Hom}_{\mathbb{F}}(J_D, A) \subset A(K)$ which agrees with ψ on $J_D[n](\overline{\mathbb{F}})$. Then the sequence $r(\phi_n)$ converges in $A(\mathbb{A}_{K, \mathbb{F}})$ to the adelic point corresponding to ψ . □

Corollary 5.2. *Suppose $\psi : D(\overline{\mathbb{F}}) \rightarrow A(\overline{\mathbb{F}})$ is the map corresponding to a point $(x_v) \in A(\mathbb{A}_{K, \mathbb{F}})^{\text{Br}}$. Then ψ is either constant, in which case $(x_v) \in A(\mathbb{F})$, or ψ has infinite image.*

Proof. It is enough to show that the image of the induced map $\psi : J_D(\overline{\mathbb{F}}) \rightarrow A(\overline{\mathbb{F}})$ has infinite image. This follows since $J_D(\overline{\mathbb{F}})$ is a divisible group and, hence, has no nontrivial finite homomorphic image. □

Corollary 5.3. *Suppose C/\mathbb{F} is a smooth proper curve with a fixed embedding $C \rightarrow J$. There is a bijection*

$$C(\mathbb{A}_{K, \mathbb{F}})^{\text{Br}} \leftrightarrow \{(\psi, P) \in \text{Hom}_{G_{\overline{\mathbb{F}}}}(J_D(\overline{\mathbb{F}}), J(\overline{\mathbb{F}})) \times J(\mathbb{F}) : \psi(D(\overline{\mathbb{F}})) \subset C(\overline{\mathbb{F}}) - P\} .$$

Proof. We have $C(\mathbb{A}_{K, \mathbb{F}})^{\text{Br}} = C(\mathbb{A}_{K, \mathbb{F}}) \cap J(\mathbb{A}_{K, \mathbb{F}})^{\text{Br}}$ by Theorem 4.2 and the definition of $C(\mathbb{A}_{K, \mathbb{F}})^{\text{ét-isog}}$. A pair (ψ, P) as in Theorem 5.1 corresponds to a reduced adelic point on C if and only if $\psi(D(\overline{\mathbb{F}})) + P \subset C(\overline{\mathbb{F}})$. □

Remark 5.4. *Given a reduced adelic point $(x_v) \in C(\mathbb{A}_{K, \mathbb{F}})^{\text{Br}}$ one can always choose an embedding of C into J such that the corresponding pair (ψ, P) has $P = 0$.*

Remark 5.5. *Zilber [Zil14, Zil], resolving a conjecture of Bogomolov, Korotiaev and Tschinkel [BKT10], has shown that if $\psi : J_D(\overline{\mathbb{F}}) \rightarrow J(\overline{\mathbb{F}})$ is an isomorphism such that $\psi(D(\overline{\mathbb{F}})) =$*

$\psi(C(\overline{\mathbb{F}}))$, then ψ is a morphism of curves composed with a limit of Frobenius maps. In particular, the corresponding reduced adelic point lies in the closure of the image of $C(K)$ in $C(\mathbb{A}_{K,\mathbb{F}})$.

Corollary 5.6. *Suppose $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}}$ and J has only finitely many abelian subvarieties. Then the induced $\psi : J_D(\overline{\mathbb{F}}) \rightarrow J(\overline{\mathbb{F}})$ is either constant or surjective, in which case J is an isogeny factor of J_D .*

Proof. By functoriality of the Brauer pairing we have $(x_v) \in J(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}}$. Now $J(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}} = J(K) \cap J(A_{K,\mathbb{F}})$ in $J(\mathbb{A}_K)$ by Proposition 2.2, so $(x_v) = \lim \phi_n : D \rightarrow J$ for some sequence of $\phi_n \in J(K)$. It suffices to show that the induced maps $\phi_n : J_D \rightarrow J$ are eventually surjective. If $\phi_n : J_D \rightarrow J$ is not surjective, then $\phi_n(D)$ is contained in a translate of a proper abelian subvariety. The intersection $C \cap (x + A)$ of $C \subset J$ with a translate of a proper abelian subvariety A is finite. If there are only finitely many proper abelian subvarieties, then these intersection numbers are bounded. But then so is $\phi_n(D) \cap C$. This implies $\psi(D(\overline{\mathbb{F}}))$ is finite, so ψ is constant by Corollary 5.2. \square

Proof of Theorem 1.5. Note that the equality $C(K) = C(\mathbb{F})$ is trivial and that it then follows from Theorem 1.2 that $C(\mathbb{A}_K)^{\text{Br}} = C(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}}$. Hence, it is enough to show that $C(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}} = C(\mathbb{F})$. Suppose $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})^{\text{Br}}$ and let $\psi : J_D(\overline{\mathbb{F}}) \rightarrow J(\overline{\mathbb{F}})$ be the corresponding homomorphism as in Corollary 5.3. We may choose embeddings $D \subset J_D$ and $C \subset J$, such that ψ restricted to $D(\overline{\mathbb{F}})$ is the map corresponding to (x_v) . Let $I = \psi(J_D(\overline{\mathbb{F}}))$. Since $J_D(\overline{\mathbb{F}})$ is generated by the divisors of degree $g = g_D$ on D , we have $I \subset W^g(C)$, where $W^g(C)$ is the image of the g -th symmetric power of C in J under the map induced by the embedding $C \rightarrow J$. As I is a topological subgroup of $J(\overline{\mathbb{F}})$ with the Zariski topology, its Zariski closure \overline{I} is an algebraic subgroup of J contained in $W^g(C)$. Since $g = \dim W^g(C) < \dim J$, we have that $C \cap \overline{I}$ is finite. As this intersection contains $\psi(D(\overline{\mathbb{F}}))$ we conclude by applying Corollary 5.2. \square

ACKNOWLEDGEMENTS

The authors were supported by the Marsden Fund Council administered by the Royal Society of New Zealand. Part of this research was carried out during the trimester ‘‘Reinventing Rational Points’’ at the Institut Henri Poincar e (IHP) and the authors would like to thank the IHP and the organizers of the trimester. They would also like to thank Damian R ossler for discussions related to the topic of this paper.

6. REFERENCES

- [AM76] Michael Artin and James S. Milne, *Duality in the flat cohomology of curves*, Invent. Math. **35** (1976), 111–129. [↑5](#)
- [BKT10] Fedor Bogomolov, Mikhail Korotiaev, and Yuri Tschinkel, *A Torelli theorem for curves over finite fields*, Pure Appl. Math. Q. **6** (2010), no. 1, Special Issue: In honor of John Tate., 245–294. [↑8](#)
- [BS08] Nils Bruin and Michael Stoll, *Deciding existence of rational points on curves: an experiment*, Experiment. Math. **17** (2008), no. 2, 181–189. [↑1](#)
- [BV95] Alexandru Buium and Jos e Felipe Voloch, *Reduction of the Manin map modulo p* , J. Reine Angew. Math. **460** (1995), 117–126. [↑5](#)
- [CVV18] Brendan Creutz, Bianca Viray, and Jos e Felipe Voloch, *The d -primary Brauer-Manin obstruction for curves*, Res. Number Theory **4** (2018), no. 2, Art. 26, 16. [↑2](#), [4](#), [6](#), [7](#)

- [GAT12] Cristian D. González-Avilés and Ki-Seng Tan, *On the Hasse principle for finite group schemes over global function fields*, Math. Res. Lett. **19** (2012), no. 2, 453–460. [↑6](#)
- [Mil17] J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field. [↑6, 8](#)
- [Mil68] ———, *The Tate-Šafarevič group of a constant abelian variety*, Invent. Math. **6** (1968), 91–105. [↑2, 7](#)
- [Poo06] Bjorn Poonen, *Heuristics for the Brauer-Manin obstruction for curves*, Experiment. Math. **15** (2006), no. 4, 415–420. [↑1](#)
- [PV10] Bjorn Poonen and José Felipe Voloch, *The Brauer-Manin obstruction for subvarieties of abelian varieties over function fields*, Ann. of Math. (2) **171** (2010), no. 1, 511–532. [↑1, 2, 4](#)
- [Rösa] Damian Rössler, *On the group of purely inseparable points of an abelian variety defined over a function field of positive characteristic II*. preprint. [↑1](#)
- [Rösb] ———, *Le groupe de Selmer des isogénies de hauteur un*. preprint. [↑5](#)
- [Sch99] Victor Scharaschkin, *Local-global problems and the Brauer-Manin obstruction*, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)—University of Michigan. [↑1, 2](#)
- [Sto07] Michael Stoll, *Finite descent obstructions and rational points on curves*, Algebra Number Theory **1** (2007), no. 4, 349–391. [↑2, 4](#)
- [Tat66] John Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144. [↑8](#)
- [WM71] W. C. Waterhouse and J. S. Milne, *Abelian varieties over finite fields*, 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York, Stony Brook, N.Y., 1969), 1971, pp. 53–64. [↑8](#)
- [Zil14] Boris Zilber, *A curve and its abstract Jacobian*, Int. Math. Res. Not. IMRN **5** (2014), 1425–1439. [↑8, 10](#)
- [Zil] ———, *A curve and its abstract Jacobian*. Corrected version of [Zil14], preprint. [↑8](#)

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CANTERBURY, PRIVATE BAG 4800, CHRISTCHURCH 8140, NEW ZEALAND

E-mail address: brendan.creutz@canterbury.ac.nz

URL: <http://www.math.canterbury.ac.nz/~b.creutz>

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CANTERBURY, PRIVATE BAG 4800, CHRISTCHURCH 8140, NEW ZEALAND

E-mail address: felipe.voloch@canterbury.ac.nz

URL: <http://www.math.canterbury.ac.nz/~f.voloch>