ETALE DESCENT OBSTRUCTION AND ANABELIAN GEOMETRY OF CURVES OVER FINITE FIELDS

BRENDAN CREUTZ AND JOSÉ FELIPE VOLOCH

Abstract. Let C and D be smooth, proper and geometrically integral curves over a finite field $\mathbb{F}$. Any morphism $D \to C$ induces a morphism of étale fundamental groups $\pi_1(D) \to \pi_1(C)$. The anabelian philosophy proposed by Grothendieck suggests that, when C has genus at least 2, all open homomorphisms between the étale fundamental groups should arise in this way from a nonconstant morphism of curves. We relate this expectation to the arithmetic of the curve $C_K := C \times_{\mathbb{F}} K$ over the global function field $K = \mathbb{F}(D)$. Specifically, we show that there is a bijection between the set of conjugacy classes of well-behaved morphism of fundamental groups and locally constant adelic points of C_K that survive étale descent. We use this to provide further evidence for the anabelian conjecture by relating it to another recent conjecture by Sutherland and the second author.

1. Introduction

Let C and D be smooth, proper and geometrically integral curves over a finite field $\mathbb{F}$. We consider the arithmetic of the curve $C_K := C \times_{\mathbb{F}} K$ over the global function field $K := \mathbb{F}(D)$. The set of adelic points surviving étale descent, denoted $C(\mathbb{A}_K)^{\text{ét}}$ and defined in Section 2.2, is a closed subset of $C(\mathbb{A}_K)$ containing the set $C(K)$ of K-rational points. If $C(\mathbb{A}_K)^{\text{ét}} \neq C(\mathbb{A}_K)$ then weak approximation must fail. It is known that not all failures of weak approximation can be explained in this way: it can happen that the topological closure of $C(K)$ is a proper subset of $C(\mathbb{A}_K)^{\text{ét}}$ because there are further obstructions arising from torsors under finite group schemes over K that are not étale [CV22, Prop 4.5]. Despite this, it is still expected that the information obtained from étale torsors should determine the set of rational points, as we now describe.

The locally constant adelic points $C(\mathbb{A}_{K,\mathbb{F}})$ form a closed subset of $C(\mathbb{A}_K)$ admitting a continuous retraction $r : C(\mathbb{A}_K) \to C(\mathbb{A}_{K,\mathbb{F}})$. See Section 2 for the precise definition. We conjecture the following.

Conjecture 1.1. *Let $\overline{C(K)}$ denote the topological closure of $C(K)$ inside $C(\mathbb{A}_K)$. Then $r(\overline{C(K)}) = C(\mathbb{A}_{K,\mathbb{F}}) \cap C(\mathbb{A}_K)^{\text{ét}}$. In particular, $C(\mathbb{A}_{K,\mathbb{F}}) \cap C(\mathbb{A}_K)^{\text{ét}} = C(\mathbb{F})$ if and only if there are no nonconstant morphisms $D \to C$.*

Conjecture 1.1 is an analogue of a conjecture in the number field case by Poonen [Poo06], in a setup first studied in [Sch99]. Conjecture 1.1 is known to be true when C has genus ≤ 1 (in which case it follows from the Tate conjecture for abelian varieties over finite fields), when the genera of C and D satisfy $g(D) < g(C)$ [CV22], and in some other cases where $C(K) = C(\mathbb{F})$ [CVV18, Theorem 2.14]. The goal of this paper is to provide further evidence for this conjecture, by relating it to anabelian geometry.

Fix geometric points $\overline{x} \in C(\overline{\mathbb{F}})$, $\overline{y} \in D(\overline{\mathbb{F}})$ where $\overline{\mathbb{F}}$ denotes an algebraic closure of $\mathbb{F}$ and let $\pi_1(C) := \pi_1(C, \overline{x})$ and $\pi_1(D) := \pi_1(D, \overline{y})$ be the étale fundamental groups of C and D with these base points. Any morphism of curves $D \to C$ induces a morphism of étale fundamental groups $\pi_1(D) \to \pi_1(C)$ up to conjugation by an element of the geometric fundamental group $\pi_1(\overline{C}) := \pi_1(C \times_{\mathbb{F}} \overline{\mathbb{F}}, \overline{x})$. Grothendieck's anabelian philosophy suggests that, when C has genus at least 2, all open homomorphisms between the étale fundamental groups should arise in this way from a nonconstant morphism of schemes [ST09, ST11]. Our main result is the following theorem which relates this expectation to Conjecture 1.1. See Definition 3.1 for the notion of well-behaved morphism.

Theorem 1.2 (Theorem 3.8)**.** *There is a bijection (explicitly constructed in the proof) between the set* $\mathrm{Hom}^{wb}_{\pi_1(\overline{C})}(\pi_1(D), \pi_1(C))$ *of well-behaved morphisms of fundamental groups up to* $\pi_1(\overline{C})$*-conjugation and the set* $C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}}$ *of locally constant adelic points surviving étale descent.*

This theorem is a strengthening of an analogous result for curves over number fields, which shows that an adelic point surviving étale descent gives rise to a section of the fundamental exact sequence [HS12, Sto07]. Combining Theorem 1.2 with the results in [CV22] we prove the following.

Theorem 1.3. *If the Jacobian J_C of C is not an isogeny factor of J_D, then Conjecture 1.1 holds for C and D.*

In addition to providing further evidence for the conjecture, this allows us to relate it in the case $g(D) = g(C)$ to a recent conjecture of Sutherland and the second author [SV19], which we now recall. We embed C into its Jacobian J_C by a choice of divisor of degree one (which always exists by the Lang-Weil estimates since C is defined over a finite field). The Hilbert class field is defined as follows. Let $\Phi : J_C \to J_C$ denote the $\mathbb{F}$-Frobenius map. Define $H(C) := (I - \Phi)^*(C) \subset J_C$, where I denotes the identity map on J. Then $H(C)$ is an unramified abelian cover of C with Galois group $J_C(\mathbb{F})$, well defined up to a twist that corresponds to a choice of divisor of degree one embedding C into J_C. Define $H_0(C) := C$, $H_1(C) := H(C)$ and successively define $H_{n+1}(C) := H_n(H(C))$ for integers $n \geq 1$.

Conjecture 1.4 ([SV19, Conjecture 2.2])**.** *Let C, D be smooth projective curves of equal genus at least 2 over a finite field $\mathbb{F}$. If, for each n, there are choices of twists such that the L-function of $H_n(C)$ is equal to the L-function of $H_n(D)$ for all $n \geq 0$, then C is isomorphic to a conjugate of D.*

Theorem 1.5. *Suppose $g(C) = g(D) \geq 2$ and assume Conjecture 1.4. Then $C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}} \cap C(\mathbb{A}_K)^{\text{ét}} \neq C(\mathbb{F})$ if and only if there is a nonconstant morphism $D \to C$.*

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2. Notation and preliminaries

2.1. **Notation.** The set of places of the global field $K = \mathbb{F}(D)$ is in bijection with the set D^1 of closed points of D. Given $v \in D^1$ we use K_v, $\mathcal{O}_v$ and $\mathbb{F}_v$ to denote the corresponding completion, ring of integers and residue field, respectively. Fix a separable closure K^{s} of K and let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$ inside K^{s}. For each $v \in D^1$, fix a separable closure K_v^{s} of K_v and an embedding $K^{\mathrm{s}} \hookrightarrow K_v^{\mathrm{s}}$. This determines an embedding $\mathbb{F}_v \subset \overline{\mathbb{F}}$

and an inclusion $\theta_v : \mathrm{Gal}(K_v) \to \mathrm{Gal}(K)$. The embedding $\mathbb{F}_v \subset \overline{\mathbb{F}}$ fixes a geometric point $\overline{v} \in D(\overline{\mathbb{F}})$ in the support of the closed point $v \in D$. The inclusions $\mathbb{F} \subset \mathbb{F}_v \subset \mathcal{O}_v \subset K_v$ endow $\mathcal{O}_v, K_v$ and the adele ring $\mathbb{A}_K$ with the structure of $\mathbb{F}$-algebra. We define the locally constant adele ring $\mathbb{A}_{K,\mathbb{F}} := \prod_{v \in D^1} \mathbb{F}_v$. This is an $\mathbb{F}$-subalgebra of the adele ring $\mathbb{A}_K$.

The constant curve $C_K = C \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(K)$ spreads out to a smooth proper model $C \times_{\mathrm{Spec}(\mathbb{F})} D$ over D. For any $v \in D^1$, this gives a reduction map $r_v : C(K_v) \to C(\mathbb{F}_v)$. Since C is proper, $C(\mathbb{A}_K) = \prod C(K_v)$ and the reduction maps give rise to a continuous projection $r : C(\mathbb{A}_K) \to C(\mathbb{A}_{K,\mathbb{F}})$ sending (x_v) to $(r_v(x_v))$.

Any locally constant adelic point $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})$ determines a unique Galois equivariant map of sets $\psi : D(\overline{\mathbb{F}}) \to C(\overline{\mathbb{F}})$ with the property that $\phi(\overline{v}) = x_v$. This induces a bijection $C(\mathbb{A}_{K,\mathbb{F}}) \leftrightarrow \mathrm{Map}_{G_{\mathbb{F}}}(D(\overline{\mathbb{F}}), C(\overline{\mathbb{F}}))$. Moreover, a locally constant adelic point on C determines, and is uniquely determined by, a map $f : D^1 \to C^1$ together with an embedding $\mathbb{F}_{f(v)} \subset \mathbb{F}_v$ for each $v \in D^1$ (See [CV22, Lemma 2.1]).

Lemma 2.1. *The composition $C(K) \to C(\mathbb{A}_K) \xrightarrow{r} C(\mathbb{A}_{K,\mathbb{F}})$ is injective. Composing this with the map $C(\mathbb{A}_{K,\mathbb{F}}) \to \mathrm{Map}(D^1, C^1)$ induces an injective map $C(K)/F \to \mathrm{Map}(D^1, C^1)$, where $C(K)/F$ denotes the set of K-rational points up to Frobenius twist, i.e., $P \sim Q$ iff there are $m, n \geq 0$ such that $F^m P = F^n Q$.*

Proof. The first statement follow from the fact (e.g., [GW10, Exercise 5.17]) that a morphism defined on a geometrically reduced variety is determined by what it does to geometric points. For the second statement see [Sti02, Proposition 2.3]. $\square$

$C(K)/F$ is finite by the theorem of de Franchis [Lan83, pg 223-224]. Over a finite field $\mathbb{F}$, there is a simpler proof. The degree of a separable map $D \to C$ is bounded by Riemann-Hurwitz. Looking now at coordinates of an embedding of C, it now suffices to show that there are only finitely many functions on $D/\mathbb{F}$ of degree bounded by some m. The zeros and poles of such a function have degree at most m over $\mathbb{F}$ so there are only finitely choices for the divisor of such a function. Finally, the function itself is determined up to a scalar in $\mathbb{F}^*$ by its divisor, but $\mathbb{F}^*$ is finite by hypothesis.

2.2. **Etale descent obstruction.** Let $f : C' \to C_K$ be a torsor under a finite étale group scheme G/K. We use $\mathrm{H}^1(K, G)$ to denote the étale cohomology set parameterizing isomorphism classes of G-torsors over K (and similarly with K replaced by $K_v, \mathcal{O}_v, \mathbb{F}_v$, etc.). The distinguished element of this pointed set is represented by the trivial torsor.

Following the terminology in [Sto07], we say an adelic point $(x_v) \in C(\mathbb{A}_K)$ survives f if the element of $\prod_v \mathrm{H}^1(K_v, G)$ given by evaluating f at (x_v) lies in the image of the diagonal map

$$\mathrm{H}^1(K, G) \xrightarrow{\prod \theta_v^*} \prod_{v \in D^1} \mathrm{H}^1(K_v, G)\,.$$

Equivalently (x_v) survives f if and only if (x_v) lifts to an adelic point on some twist of f by a cocycle representing a class in $\mathrm{H}^1(K, G)$. We use $C(\mathbb{A}_K)^{\text{ét}}$ to denote the set of adelic points surviving all C-torsors under étale group schemes over K. Then $C(\mathbb{A}_K)^{\text{ét}}$ is a closed subset of $C(\mathbb{A}_K)$ containing $C(K)$. We define $C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}} = C(\mathbb{A}_K)^{\text{ét}} \cap C(\mathbb{A}_{K,\mathbb{F}})$. By [CV22, Proposition 4.6] an adelic point lies in $C(\mathbb{A}_K)^{\text{ét}}$ if and only if its image under the reduction map $r : C(\mathbb{A}_K) \to C(\mathbb{A}_{K,\mathbb{F}})$ lies in $C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}}$.

The following lemma is a special case of a well known statement in étale cohomology over a henselian ring (cf. [Mil80, Remark 3.11(a) on p. 116]).

Lemma 2.2. *For an étale group scheme G over $\mathbb{F}$ we have $\mathrm{H}^1(\mathcal{O}_v, G) = \mathrm{H}^1(\mathbb{F}_v, G)$.*

Proof. The canonical surjection $q\colon \mathcal{O}_v \to \mathbb{F}_v$ induces a map $q_*\colon \mathrm{H}^1(\mathcal{O}_v, G) \to \mathrm{H}^1(\mathbb{F}_v, G)$. This map is injective by Hensel's lemma. On the other hand, the inclusion $i\colon \mathbb{F}_v \to \mathcal{O}_v$ satisfies $q \circ i = \mathrm{id}$. It follows that q_* must also be surjective. □

An element of $\mathrm{H}^1(K_v, G)$ is called unramified if it lies in the image of the map $\mathrm{H}^1(\mathcal{O}_v, G) \to \mathrm{H}^1(K_v, G)$ induced by the inclusion $\mathcal{O}_v \subset K_v$. Thus, the lemma identifies $\mathrm{H}^1(\mathbb{F}_v, G)$ with the set of unramified elements in $\mathrm{H}^1(K_v, G)$.

3. Connection to anabelian geometry

Fix a base point $\overline{x} : \operatorname{Spec} \overline{\mathbb{F}} \to \overline{D} := D \times_{\operatorname{Spec}(\mathbb{F})} \operatorname{Spec}(\overline{\mathbb{F}})$. Composing with the canonical maps $\overline{D} \to D$ and $D \to \operatorname{Spec}(\mathbb{F})$, this serves as well to fix base points of D and $\operatorname{Spec}(\mathbb{F})$. The basepoint of $\operatorname{Spec}(\mathbb{F})$ agrees with that determined by the algebraic closure $\mathbb{F} \subset \overline{\mathbb{F}}$ fixed above. This leads to the fundamental exact sequence

$$1 \to \pi_1(\overline{D}) \to \pi_1(D) \to \mathrm{Gal}(\mathbb{F}) \to 1\,, \tag{3.1}$$

where $\pi_1(-)$ denotes the étale fundamental group with base point as chosen above. A choice of base point $\operatorname{Spec} \overline{\mathbb{F}} \to \overline{C}$ determines a similar sequence for C.

The choice of separable closure of K identifies $\pi_1(D)$ with the Galois group of the maximal extension K^{unr} of K which is everywhere unramified. For each closed point $v \in D^1$, The embedding $\theta_v : \mathrm{Gal}(K_v) \to \mathrm{Gal}(K^{\mathrm{s}})$ induces a section map $t_v : \mathrm{Gal}(\mathbb{F}_v) \simeq \mathrm{Gal}(K_v^{\mathrm{unr}}|K_v) \to \pi_1(D)$ whose image is a decomposition group $T_v \subset \pi_1(D)$ above v.

Definition 3.1. *A continuous morphism $\pi_1(D) \to \pi_1(C)$ is well-behaved if every decomposition group of $\pi_1(D)$ is mapped to an open subgroup of a decomposition group of $\pi_1(C)$. Let $\mathrm{Hom}^{\mathrm{wb}}(\pi_1(D), \pi_1(C))$ denote the set of well-behaved homomorphisms of profinite groups and for a subgroup $H < \pi_1(C)$ let $\mathrm{Hom}^{\mathrm{wb}}_H(\pi_1(D), \pi_1(C))$ denote the quotient of $\mathrm{Hom}^{\mathrm{wb}}(\pi_1(D), \pi_1(C))$ by the action given by composition with an inner automorphism of $\pi_1(C)$ coming from an element of H.*

Remark 3.2. *Here is an example of a poorly behaved homomorphism. Suppose the genus of C is at least 2. By [Sti13, Theorem 226] there are uncountably many sections $\mathrm{Gal}(\mathbb{F}) \to \pi_1(C)$ that are not conjugate to any section coming from a point in $C(\mathbb{F})$. Composing such a section with the canonical surjection $\pi_1(D) \to \mathrm{Gal}(\mathbb{F})$ gives a continuous morphism $\pi_1(D) \to \pi_1(C)$ that is not well-behaved.*

Proposition 3.3. *Suppose $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}}$. For each $v \in D^1$, let $S_v \subset \pi_1(C_{\mathbb{F}_v}) \subset \pi_1(C)$ be a decomposition group above the closed point $x_v \in C_{\mathbb{F}_v}$. Then there exists a well-behaved homomorphism $\phi : \pi_1(D) \to \pi_1(C)$ inducing a morphism of exact sequences*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\overline{D}) \arrow[r] \arrow[d, "\phi"] & \pi_1(D) \arrow[r] \arrow[d, "\phi"] & \mathrm{Gal}(\mathbb{F}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(\overline{C}) \arrow[r] & \pi_1(C) \arrow[r] & \mathrm{Gal}(\mathbb{F}) \arrow[r] & 1
\end{tikzcd}$$

such that, for each $v \in D^1$, there exists $\gamma_v \in \pi_1(\overline{C})$ such that $\phi(T_v) = \gamma_v(S_v)\gamma_v^{-1}$.

Proof. For each $v \in D^1$ the choice of decomposition group $S_v \subset \pi_1(C_{\mathbb{F}_v})$ above x_v determines a section map $s_v : \mathrm{Gal}(\mathbb{F}_v) \to \pi_1(C_{\mathbb{F}_v}) \subset \pi_1(C)$ with image S_v. For any finite continuous quotient $\rho_G : \pi_1(C) \to G$, the composition $\rho_G \circ s_v : \mathrm{Gal}(\mathbb{F}_v) \to G$ determines a class in $\mathrm{H}^1(\mathbb{F}_v, G) = \mathrm{Hom}_G(\mathrm{Gal}(\mathbb{F}_v), G)$, the group of homomorphisms up to G-conjugation. Here we view G as a constant group scheme over $\mathbb{F}$. By Lemma 2.2 we may view $\mathrm{H}^1(\mathbb{F}_v, G)$ as a subgroup of $\mathrm{H}^1(K_v, G)$. In terms of descent, ρ_G corresponds to a torsor in $\mathrm{H}^1(C, G) = \mathrm{H}^1(\pi_1(C), G) = \mathrm{Hom}_G(\pi_1(C), G)$ and $\rho_G \circ s_v$ is the evaluation of this torsor at $x_v \in C(\mathbb{F}_v)$. So the fact that (x_v) survives étale descent implies that there is a global class $s \in \mathrm{H}^1(K, G)$ such that for all $v \in D^1$, $\theta_v^*(s) = \rho_G \circ s_v$ in $\mathrm{H}^1(\mathbb{F}_v, G) \subset \mathrm{H}^1(K_v, G)$. Note that such an s must lie in (the image under inflation of) the group $\mathrm{H}^1(\pi_1(D), G) = \mathrm{Hom}_G(\pi_1(D), G)$, since the s_v are all unramified.

For each $v \in D^1$, the condition $\theta_v^*(s) = \rho_G \circ s_v \in \mathrm{H}^1(K_v, G)$ is equivalent to $s \circ t_v = \rho_G \circ s_v$ in $\mathrm{H}^1(\mathbb{F}_v, G) = \mathrm{Hom}_G(\mathrm{Gal}(\mathbb{F}_v), G)$. Let $G_v = \rho_G(\pi_1(C_{\mathbb{F}_v})) \subset G$ be the image of ρ_G restricted to the normal subgroup $\pi_1(C_{\mathbb{F}_v})$. Then G_v is normal in G and contains the image of $\rho_G \circ s_v$, so it must also contain the image of $s \circ t_v$. Since G is constant, the map $\mathrm{H}^1(\mathbb{F}_v, G_v) \to \mathrm{H}^1(\mathbb{F}_v, G)$ induced by the inclusion $G_v \subset G$ is injective. It follows that $\rho_G \circ s_v$ and $s \circ t_v$ are equal as elements of $\mathrm{H}^1(\mathbb{F}_v, G_v)$.

By the Borel-Serre Theorem (see [Poo17, 5.12.29]) the fibers of the map $\mathrm{H}^1(K, G) \to \prod_{v \in D^1} \mathrm{H}^1(K_v, G)$ are finite. It follows that the set

$$S_G := \{s' : \pi_1(D) \to G \mid \forall\, v \in D^1,\ s' \circ t_v = \rho_G \circ s_v \text{ in } \mathrm{H}^1(\mathbb{F}_v, G_v)\}$$

is finite, and it is nonempty by the discussion above. As in the proof of [HS12, Proposition 1.2] it follows that the inverse limit over G of these sets is nonempty. An element of $\varprojlim S_G$ is a homomorphism $\phi : \pi_1(D) \to \varprojlim G = \pi_1(C)$ with the property that for all $v \in D^1$, the maps $\phi \circ t_v$ and s_v are conjugate by an element of $\pi_1(C_{\mathbb{F}_v}) = \varprojlim G_v$. We claim that $\phi \circ t_v$ and s_v are in fact $\pi_1(\overline{C})$-conjugate. To see this, let $p : \pi_1(C) \to \mathrm{Gal}(\mathbb{F})$ be the canonical surjection. Suppose $\gamma_v \in \pi_1(C_{\mathbb{F}_v})$ conjugates s_v to $\phi \circ t_v$. We claim $\gamma_v' := \gamma_v \cdot s_v(p(\gamma_v^{-1})) \in \pi_1(\overline{C})$ and conjugates s_v to $\phi \circ t_v$. (Note that $s_v(p(\gamma_v^{-1}))$ makes sense as $p(\gamma_v) \in \mathrm{Gal}(\mathbb{F}_v)$.) To see that $\gamma_v' \in \pi_1(\overline{C})$ we use that $p \circ s_v$ is the identity map on $\mathrm{Gal}(\mathbb{F}_v)$ to compute

$$p(\gamma_v') = p\left(\gamma_v \cdot s_v(p(\gamma_v^{-1}))\right) = p(\gamma_v) \cdot (p \circ s_v)(p(\gamma^{-1})) = p(\gamma)p(\gamma^{-1}) = 1\,.$$

To see that γ_v' conjugates s_v to $\phi \circ t_v$ we compute, for arbitrary $\sigma \in \mathrm{Gal}(\mathbb{F}_v)$,

$$\begin{aligned}
\gamma_v' \cdot s_v(\sigma) \cdot \gamma_v'^{-1} &= [\gamma_v \cdot s_v(p(\gamma_v^{-1}))] \cdot s_v(\sigma) \cdot [\gamma_v \cdot s_v(p(\gamma_v^{-1}))]^{-1} \\
&= \gamma_v \cdot s_v(p(\gamma^{-1})\sigma p(\gamma)) \cdot \gamma_v^{-1} \\
&= \gamma_v \cdot s_v(\sigma) \cdot \gamma_v^{-1}\,,
\end{aligned}$$

where the final equality uses that $\mathrm{Gal}(\mathbb{F}_v)$ is abelian.

Finally, let us show that ϕ induces a morphism of exact sequences as in the statement. Write $p_D : \pi_1(D) \to \mathrm{Gal}(\mathbb{F})$ for the canonical map and use p_C similarly. Since $p_C \circ s_v$ is the identity on the abelian group $\mathrm{Gal}(\mathbb{F}_v)$, for any $\sigma \in \mathrm{Gal}(\mathbb{F}_v)$ we have

$$p_C(\phi(t_v(\sigma))) = p_C(\gamma_v \cdot s_v(\sigma) \cdot \gamma_v^{-1}) = p_C(s_v(\sigma)) = \sigma\,.$$

So for any $x \in \pi_1(D)$ whose image under p_D lies in $\mathrm{Gal}(\mathbb{F}_v)$ we have $p_D(x) = p_C(\phi(x))$. As this holds for all $v \in D^1$, we must have $p_D = p_C \circ \phi$. So ϕ induces a morphism of exact sequences as stated. ∎

Remark 3.4. *The construction of the morphism ϕ in the preceding proof is similar to the proof of* [HS12, Proposition 1.1]. *However, the verification that it interpolates the s_v up to conjugation in $\pi_1(\overline{C})$ rather than just in $\pi_1(C)$ is necessarily different from the approach in* [HS12, Proposition 1.2].

Construction 3.5. Let $\phi : \pi_1(D) \to \pi_1(C)$ be a well-behaved homomorphism. From this we construct a locally constant adelic point $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})$ as follows. Let $\tilde{D}$ and $\tilde{C}$ denote the universal covers of D and C. The decomposition groups of $\pi_1(D)$ and $\pi_1(C)$ correspond to closed points on $\tilde{D}$ and $\tilde{C}$. As we have assumed C to be hyperbolic, the intersection of any two distinct decomposition groups of $\pi_1(C)$ is open in neither (see for example [ST11, Proposition 1.5]). So the well-behaved map ϕ determines a map $\tilde{\phi}\colon \tilde{D}^1 \to \tilde{C}^1$ by declaring $\tilde{\phi}(\tilde{v})$ to be the point of $\tilde{C}$ whose corresponding decomposition group contains $\phi(\mathcal{D}_{\tilde{v}})$. Given a closed point $v \in D^1$, the embedding $\theta_v : \mathrm{Gal}(K_v) \to \mathrm{Gal}(K)$ determines a decomposition group T_v above v and consequently a pro-point $\tilde{v} \in \tilde{D}$. Define $x_v \in C(\mathbb{F}_v) = C_{\mathbb{F}_v}(\mathbb{F}_v)$ to be the image of $\tilde{\phi}(\tilde{v})$ on $C_{\mathbb{F}_v}$. Ranging over the closed points of D, this determines a locally constant adelic point $(x_v) \in \prod_{v \in D^1} C(\mathbb{F}_v) = C(\mathbb{A}_{K,\mathbb{F}})$.

Remark 3.6. *Note that $\pi_1(C)$ acts on the set of pro-points $\tilde{w}$ above a given $w \in C^1$ and that any two pro-points above $w \in C^1$ in the same $\pi_1(\overline{C})$-orbit have the same image on $C_{\mathbb{F}_w}$. It follows that the adelic point (x_v) constructed in 3.5 depends on ϕ only up to $\pi_1(\overline{C})$ conjugacy. Similarly, the image of (x_v) in $\mathrm{Map}(D^1, C^1)$ under the map in Lemma 2.1 depends on ϕ only up to $\pi_1(C)$-conjugacy.*

Lemma 3.7. *Suppose $\phi \in \mathrm{Hom}^{\mathrm{wb}}_{\pi_1(\overline{C})}(\pi_1(D), \pi_1(C))$ and let $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})$ be the locally constant adelic point given by Construction 3.5. Then $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}}$.*

Proof. For $v \in D^1$, let $t_v : \mathrm{Gal}(\mathbb{F}_v) \to \pi_1(D)$ be the section map as defined at the beginning of this section. Define $s_v = \phi \circ t_v : G_{\mathbb{F}_v} \to \pi_1(C)$. By construction, the image of s_v is a decomposition group of $\pi_1(C)$ above $x_v \in C(\mathbb{F}_v)$. Let $\alpha\colon C' \to C$ be a torsor under a finite group scheme $G/\mathbb{F}$. Then α represents a class in $\mathrm{H}^1(C, G) = \mathrm{H}^1(\pi_1(C), G(\overline{\mathbb{F}}))$, where the action of $\pi_1(C)$ on $G(\overline{\mathbb{F}})$ is induced by the projection $\pi_1(C) \to \mathrm{Gal}(\mathbb{F})$. The evaluation of α at x_v is the class of $\alpha \circ s_v$ in $\mathrm{H}^1(\mathbb{F}_v, G)$. Since $\alpha \circ s_v = \alpha \circ \phi \circ t_v = t_v^*(\alpha \circ \phi)$ we see that $\alpha \circ \phi$ lies in the images of the horizontal maps in the following commutative diagram whose vertical maps come from inflation:

$$\begin{tikzcd}
\mathrm{H}^1(K, G) \arrow[r, "\theta_v^*"] & \mathrm{H}^1(K_v, G) \\
\mathrm{H}^1(\pi_1(D), G) \arrow[u, hook] \arrow[r, "t_v^*"] & \mathrm{H}^1(\mathbb{F}_v, G) \arrow[u, hook]
\end{tikzcd},$$

As this holds for all $v \in D^1$, we see that the evaluation of α at the adelic point (x_v) lies in the diagonal image of $\mathrm{H}^1(K, G)$. ∎

Theorem 3.8. *Construction 3.5 induces bijections*

$$\begin{tikzcd} C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}} \arrow[r, leftrightarrow] \arrow[d, twoheadrightarrow] & \mathrm{Hom}^{\mathrm{wb}}_{\pi_1(\overline{C})}(\pi_1(D),\pi_1(C)) \arrow[d, twoheadrightarrow] \\ \mathrm{Map}(D^1,C^1)^{\text{ét}} \arrow[r, leftrightarrow] & \mathrm{Hom}^{\mathrm{wb}}_{\pi_1(C)}(\pi_1(D),\pi_1(C))\,. \end{tikzcd}$$

where $\mathrm{Map}(D^1,C^1)^{\text{ét}}$ *denotes the image of* $C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}}$ *in* $\mathrm{Map}(D^1,C^1)$ *under the map in Lemma 2.1.*

Proof. Proposition 3.3 gives a map of sets

$$C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}} \to \mathrm{Hom}^{\mathrm{wb}}_{\pi_1(\overline{C})}(\pi_1(D),\pi_1(C))\,,$$

while Construction 3.5 and Lemma 3.7 give an injective map

$$\mathrm{Hom}^{\mathrm{wb}}_{\pi_1(\overline{C})}(\pi_1(D),\pi_1(C)) \hookrightarrow C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}}\,.$$

One easily checks that these maps are inverse to one another, so they are inverse bijections.

Surjectivity of the first vertical map is given in Lemma 2.1 and surjectivity of the other is immediate from the definition. One deduces the bijection in the bottom row from that in the top row using Remark 3.6. ☐

Proposition 3.9. *Let $\phi : \pi_1(D) \to \pi_1(C)$ be a well-behaved morphism corresponding to a locally constant adelic point surviving étale descent $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}}$ as given by Proposition 3.3. If $(x_v) \notin C(\mathbb{F})$, then ϕ has open image and the map $\psi : D(\overline{\mathbb{F}}) \to C(\overline{\mathbb{F}})$ induced by (x_v) is surjective.*

Corollary 3.10. *If $\phi : \pi_1(D) \to \pi_1(C)$ is a well-behaved homomorphism. The image of ϕ is either open or is a decomposition group above a point $v \in C(\mathbb{F})$.*

Proof. Suppose the image of ϕ is not open. Then we find a sequence of open subgroups $U_i \subset \pi_1(C)$ of index approaching infinity all of which contain the image of ϕ. By Proposition 3.3 the image of ϕ maps surjectively onto $\mathrm{Gal}(\mathbb{F})$ under the canonical map $\pi_1(C) \to \mathrm{Gal}(\mathbb{F})$. Hence, the induced maps $U_i \to \mathrm{Gal}(\mathbb{F})$ are surjective, so that the U_i correspond to geometrically connected étale coverings $C_i \to C$ of genus approaching infinity. For each we have a well-behaved homomorphism $\pi_1(D) \to U_i = \pi_1(C_i)$. By Theorem 3.8, these correspond to unobstructed adelic points $(x_v^{(i)}) \in C_i(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}}$ which lift $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})$. Eventually $g(C_i) > g(D)$, in which case [CV22, Theorems 1.2,1.3 and 1.5] imply that $C_i(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}} = C_i(\mathbb{F})$. But then $(x_v) \in C(\mathbb{F})$. Therefore, if (x_v) is nonconstant, then ϕ must have open image. In this case, the image of ϕ contains a finite index subgroup of each decomposition group. This implies that $\psi : D(\overline{\mathbb{F}}) \to C(\overline{\mathbb{F}})$ is surjective. ☐

4. Proofs of the theorems in the introduction

4.1. **Proof of Theorem 1.3.** Suppose $(x_v) \in C(\mathbb{A}_{K,\mathbb{F}})^{\text{ét}} \setminus C(\mathbb{F})$. By Proposition 3.9 the Galois equivariant map $\psi : D(\overline{\mathbb{F}}) \to C(\overline{\mathbb{F}})$ induced by (x_v) is surjective. By [CV22, Corollary 5.3] this induces a surjective $G_{\mathbb{F}}$-equivariant homomorphism $\phi_* : J_D(\overline{\mathbb{F}}) \to J_C(\overline{\mathbb{F}})$. For any $\ell \neq p$, this yields a surjective homomorphism of the ℓ-adic Tate modules of $T_\ell(J_D) \to T_\ell(J_C)$, so J_C is an isogeny factor of J_D by the Tate conjecture for abelian varieties over finite fields [Tat66].

4.2. **Proof of Theorem 1.5.** Let $x = (x_v) \in C(\mathbb{A}_K)^{\text{ét}} \setminus C(\mathbb{F})$. Since $H(C) \to C$ is an étale cover, x lifts to a twist of $H(C)$ by an element $\xi \in \mathrm{H}^1(K, J_C(\mathbb{F})) = \mathrm{Hom}(G_K, J_C(\mathbb{F}))$. Let L/K be the fixed field of $\ker(\xi)$. Then L/K is unramified since, locally, it is given as the extension generated by the roots of $(I - \Phi)(y) = x_v$, and L/K is abelian since $\mathrm{Gal}(L/K)$ is a subgroup of $J_C(\mathbb{F})$. Thus L is a subfield of the function field K' of $H(D)$ (for a suitable embedding $D \to J_D$). Viewing x as an adelic point on C over K', we have have $x \in C(\mathbb{A}_{K'})^{\text{ét}}$ by [Sto07, Proposition 5.15]. By the above this adelic point lifts to $H(C)(\mathbb{A}_{K'})^{\text{ét}}$.

From Theorem 1.3, we get that that $H(C)$ and $H(D)$ have the same L-function. Now we are in the same situation as before with $H(C), H(D)$ in place of C, D. Iterating this process we obtain towers such that $H_n(D)$ and $H_n(C)$ have the same L-functions. Assuming Conjecture 1.4 this implies $C(K) \neq C(\mathbb{F})$.

Remark 4.1. *The paper* [BV20] *proves a theorem very close in spirit to Conjecture 1.4 using L-functions with characters. It would be very desirable to have a proof of Conjecture 1.1 in the equigenus case from the main theorem of* [BV20] *along the lines of the above proof but we haven't succeeded in producing it.*

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