GALOIS INVARIANTS OF FINITE ABELIAN DESCENT AND BRAUER SETS

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Abstract. For a variety over a global field, one can consider subsets of the set of adelic points of the variety cut out by finite abelian descent or Brauer-Manin obstructions. Given a Galois extension of the ground field one can consider similar sets over the extension and take Galois invariants. In this paper, we study under which circumstances the Galois invariants recover the obstruction sets over the ground field. As an application of our results, we study finite abelian descent and Brauer-Manin obstructions for isotrivial curves over function fields and extend results obtained by the first and last authors for constant curves to the isotrivial case.

1. Introduction

Let X/k be a smooth projective and geometrically connected variety over a global field k with separable closure k^s. It is well known that the set $X(k)$ of k-rational points of X may fail to be dense in the set $X(\mathbb{A}_k) = \prod_v X(k_v)$ of adelic points of X, as there can be cohomological obstructions mediated by the Brauer group $\mathrm{Br}(X)$ and/or by the finite abelian descent obstruction [14, 19]. This remains true even if one replaces $X(\mathbb{A}_k)$ with its space $X(\mathbb{A}_k)_\bullet$ of connected components, which only differs from $X(\mathbb{A}_k)$ at the archimedean places. By abuse of language we will refer to elements of $X(\mathbb{A}_k)_\bullet$ also as adelic points. It is clear that the assignment $L \mapsto X(L)$ defines a sheaf of sets on $\mathrm{Spec}(k)_{\text{ét}}$ (i.e., an étale sheaf) in the sense that, for any finite Galois extension L/k, we have $X(L)^{\mathrm{Gal}(L/k)} = X(k)$. In this note we investigate whether the sets cut out by these cohomological obstructions also define étale sheaves in this sense.

For a torsor $f : Y \to X$ under a finite abelian algebraic group G/k, let $X(\mathbb{A}_k)_\bullet^f \subset X(\mathbb{A}_k)_\bullet$ denote the set adelic points which survive f [21, Definition 5.2]. Then $X(\mathbb{A}_k)_\bullet^f$ contains the topological closure $\overline{X(k)}$ of $X(k)$ in $X(\mathbb{A}_k)_\bullet$. For a finite separable extension L/k we use $X(\mathbb{A}_L)_\bullet^f$ to denote the set of L-adelic points on X which lift to a twist of the base changed torsor $f_L : Y_L \to X_L$. When L/k is Galois, there is a natural inclusion $X(\mathbb{A}_k)_\bullet^f \subset (X(\mathbb{A}_L)_\bullet^f)^{\mathrm{Gal}(L/k)}$. Our first main result shows that this will be an equality for all Galois extensions L/k if and only if G does not contain any nontrivial étale subgroup.

Theorem 1.1 (Theorems 2.1 and 2.3)**.** *Let X be a smooth projective and geometrically connected variety over k admitting a geometrically connected torsor $f : Y \to X$ under a finite abelian group scheme G over a global field k. Then the following are equivalent.*

1. *There exists a finite separable extension K/k and a Galois extension L/K such that $X(\mathbb{A}_K)_\bullet^f \neq (X(\mathbb{A}_L)^f)_\bullet^{\mathrm{Gal}(L/K)}$;*
2. *$G(k^s) \neq 0$.*

For subvarieties of abelian varieties with torsion free Néron-Severi group (e.g., if X is a curve or an abelian variety) we will show that the failure of finite abelian descent sets to determine étale sheaves goes away in the limit. Define $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} := \bigcap_{f:Y\to X} X(\mathbb{A}_k)_\bullet^f$, where the intersection is taken over all geometrically connected torsors under finite abelian group schemes.

Theorem 1.2. *Suppose $X \subset A$ is a subvariety of an abelian variety over a global field with torsion free Néron-Severi group. Then for any finite Galois extension L/k we have*

$$X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \left(X(\mathbb{A}_L)_\bullet^{\text{f-ab}}\right)^{\mathrm{Gal}(L/k)} .$$

When X is a subvariety of an abelian variety A/k the finite abelian descent obstruction is closely related to the Brauer-Manin obstruction. There are containments

$$(\dagger)\qquad \begin{array}{ccccc} \overline{X(k)} & \subset & X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} & \subset & X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \\ \cap & & \cap & & \cap \\ X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} & \subset & X(\mathbb{A}_k)_\bullet \cap A(\mathbb{A}_k)_\bullet^{\mathrm{Br}} & \subset & X(\mathbb{A}_k)_\bullet \cap A(\mathbb{A}_k)_\bullet^{\text{f-ab}}. \end{array}$$

For the following conjecture we refer to [19, p. 133], [21, Section 8], [13, Question 7.4] and [15, Conjecture C].

Conjecture 1.3. For X a smooth closed subvariety of an abelian variety A over a global field k all of the containments in (†) are equalities.

This conjecture has been proven when $A(k)$ and $Ш(A/k)$ are finite [17] and for most nonisotrivial coset-free subvarieties of abelian varieties over global function fields [15], including all nonisotrivial curves of genus at least 2 over global function fields [6].

It is not difficult to show that the bottom left set of (†) defines an étale sheaf (see Lemma 3.5). Thus, Theorem 1.2 was predicted by Conjecture 1.3 and can be seen as some modest evidence for it. In Section 3 we verify that the other terms in (†) define étale sheaves in a number of cases, including the case when X is a curve (See Theorem 3.9). We also show that, in general, none of the sets in the top row of (†) define étale sheaves if we consider varieties that do not embed into an abelian variety (See Propositions 3.11 and 3.13). These examples complement various recent results studying the behaviour of obstructions under base extension [1, 10, 16, 23].

Part of our motivation for studying these questions comes from the desire to extend the results of [5] concerning the Brauer-Manin obstruction for constant curves over global function fields to the case of isotrivial curves. Recall that a variety X over the function field $k = \mathbb{F}(D)$ of a smooth projective curve D over a finite field $\mathbb{F}$ is called constant if there is a variety $X_0/\mathbb{F}$ such that $X \simeq X_0 \times_{\mathbb{F}} k$. A variety X/k is called isotrivial if there is a finite extension L/k such that $X \times_k L$ is constant. We note that, unlike the nonisotrivial case, it is not immediate that $\overline{X(k)}$ is equal to the subset of $\overline{X(L)}$ fixed by Galois, as there are isotrivial curves of every genus with $X(k) \neq \overline{X(k)}$.

As an application of the results above we generalize [5, Theorems 1.1 and 1.2] to the case of isotrivial curves. See Theorems 4.13 and 4.11 below. We also deduce the following.

Theorem 1.4. *Let X be a smooth projective and isotrivial curve over a global function field k. Suppose X becomes constant after base change to L/k. Then $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$ if and only if $\overline{X(L)} = X(\mathbb{A}_L)^{\mathrm{Br}}$. Moreover, Conjecture 1.3 holds for all isotrivial curves over global function fields if it holds for all constant curves over global function fields.*

Note that we have $X(\mathbb{A}_k) = X(\mathbb{A}_k)_\bullet$ in the function field case. Some instances where the equality $\overline{X(L)} = X(\mathbb{A}_L)^{\mathrm{Br}}$ is known to hold for a constant curve over L are given in [7, Theorem 2.14] and [5, Theorem 1.5].

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2. Descent sets for torsors under finite abelian group schemes

In this section we will prove Theorem 1.1.

2.1. The nontrivial étale subgroup scheme case.

For a finite abelian group scheme G over k, there is a unique maximal étale subgroup scheme G_e. It is determined by the property that $G_e(k^s) = G(k^s)$. We begin by proving Theorem 1.1 in the case that G contains a nontrivial étale subgroup scheme.

Theorem 2.1. *Let G be a finite abelian group scheme over a global field k such that $G(k^s) \neq 0$. Let X be a smooth projective over k which admits a geometrically connected torsor $f : Y \to X$ under G. Then there exists a finite separable extension K/k and a finite Galois extension L/K such that*

$$X(\mathbb{A}_K)^f_\bullet \subsetneq \left(X(\mathbb{A}_L)^f_\bullet\right)^{\mathrm{Gal}(L/K)}.$$

In particular, the assignment $L \mapsto X(\mathbb{A}_L)^f_\bullet$ does not define a sheaf of sets on $\mathrm{Spec}(k)_{\text{ét}}$.

Lemma 2.2. *Let X/k be a smooth projective variety over a global field k and let $f : Y \to X$ be a torsor under a finite étale abelian group scheme G/k. If $Y(\mathbb{A}_k)_\bullet \neq \emptyset$, then there exists an adelic point $x \in X(\mathbb{A}_k)_\bullet$ and a finite separable extension L/k such that $x \in X(\mathbb{A}_L)^f_\bullet \setminus X(\mathbb{A}_k)^f_\bullet$.*

Proof. Call a class $\xi \in H^1(k, G)$ finitely supported if the image of ξ under the restriction map $H^1(k, G) \to H^1(k_v, G)$ is trivial for all but finitely many places v of k. Let K/k be the splitting field of G, i.e., the minimal Galois extension K/k such that $G(K) = G(\overline{k})$. By [12, I.9.3] any finitely supported class lies in the image of the inflation map $H^1(K/k, G) \to H^1(k, G)$. It follows that $H^1(k, G)$ contains only finitely many finitely supported classes. Since $H^1(k, G)$ is infinite, there are infinitely many classes that are not finitely supported.

We claim that there are infinitely many places v of k such that the map $f : Y(k_v) \to X(k_v)$ is not surjective. To see this, let $\xi \in H^1(k, G)$ be a class which is not finitely supported and let $f^\xi : Y^\xi \to X$ be the corresponding twist. Then Y^ξ is smooth and geometrically connected so has points over k_v for all but finitely many v by the Lang-Weil estimates and Hensel's lemma. A point $x \in f^\xi(Y^\xi(k_v)) \subset X(k_v)$ lies in $f(Y(k_v))$ if and only if $\mathrm{res}_v(\xi) = 0$. Since ξ is not finitely supported, there are infinitely many places where $f^\xi(Y^\xi(k_v))$ is a nonempty subset of $X(k_v)$ disjoint from $f(Y(k_v))$.

Let M be the maximum number of places at which a finitely supported class in $H^1(k, G)$ is nonzero and let $w_0, \dots, w_M$ be $M+1$ places of k such that $f : Y(k_{w_i}) \to X(k_{w_i})$ is not surjective. Let $(x_v) \in Y(\mathbb{A}_k)_\bullet$ and choose $y_{w_i} \in X(k_{w_i}) \setminus f(Y(k_{w_i}))$ for $i = 0, \dots, M$. Define $(z_v) \in X(\mathbb{A}_k)_\bullet$ by setting $z_v = f(x_v) \in X(k_v)$ for $v \notin \{w_0, \dots, w_M\}$ and setting $z_{w_i} = y_{w_i}$ for $i = 0, \dots, M$.

For all places $v \notin \{w_0, \dots, w_M\}$ we have that $z_v^* f = f(x_v)^* f = 0 \in H^1(k_v, G)$. It follows that $(z_v)^* f$ is nonzero precisely at the $M+1$ places $w_0, \dots, w_M$. From the definition of the integer M it follows that $(z_v)^* f$ does not lie in the diagonal image $H^1(k, G) \to \prod_v H^1(k_v, G)$, and so $(z_v) \in X(\mathbb{A}_k)_\bullet \setminus X(\mathbb{A}_k)^f_\bullet$.

Since G is étale we have that for each $i = 0, \dots, M$, there is a finite separable extension L^{w_i}/k_{w_i} that kills $z_{w_i}^* f$. Moreover, we can find a global Galois extension L/k such that, for all $i = 0, \dots, M$ and all primes $u | w_i$ we have that the extension L_u/k_{w_i} kills $z_{w_i}^* f$. Then $(z_v) \in X(\mathbb{A}_L)^f_\bullet \cap X(\mathbb{A}_k)_\bullet$ as required. □

Proof of Theorem 2.1. Let K/k be a finite separable extension such that $X(K) \neq \emptyset$. Then there exists a twist $f' : Y' \to X_K$ of $Y_K \to X_K$ with $Y'(K) \neq \emptyset$. By [21, Lemma 5.3(5)] we have that $X(\mathbb{A}_K)^{f'}_\bullet = X(\mathbb{A}_K)^f_\bullet$ and similarly over L, so replacing Y by this twist if necessary we may assume $Y(K) \neq \emptyset$. Let $G_e \subset G$ be the maximal étale subgroup scheme (i.e., defined by $G_e(k^s) = G(k^s)$). Then f factors as a composition of torsors $f_e : Y \to Z$ and $g : Z \to X$ under G_e and G/G_e, respectively.

Then f_e satisfies the conditions of Lemma 2.2 over K, so there is a finite Galois extension L/K and some $z \in Z(\mathbb{A}_L)^{f_e}_\bullet \cap Z(\mathbb{A}_K)_\bullet$ which does not lie in $Z(\mathbb{A}_K)^{f_e}_\bullet$. Define $x := g(z)$, which by construction lies in $g\left(Z(\mathbb{A}_L)^{f_e}_\bullet \cap Z(\mathbb{A}_K)_\bullet\right) \subset X(\mathbb{A}_L)^f_\bullet \cap X(\mathbb{A}_K)_\bullet$. To complete the proof it is enough to show that $x \notin X(\mathbb{A}_K)^f_\bullet$.

By way of contradiction, suppose $x \in X(\mathbb{A}_K)^f_\bullet \subset X(\mathbb{A}_K)^g_\bullet$. Then x lifts to a twist $f' = g' \circ f'_e$ of $f = g \circ f$ by a class in $H^1(K, G)$ and x lifts to the twist g' of g by the image ξ of this class in $H^1(K, G/G_e)$. Since x also lifts under g we must have that $\xi \in Ш^1(K, G/G_e)$. The map $G \to G/G_e$ is étale, so any separable point of the quotient will lift to a separable point of G. It follows that $(G/G_e)(k^s) = 0$. Then $Ш^1(K, G/G_e) = 0$ by [8, Main Theorem]. So $g = g'$ and we see that $x \in g(Z(\mathbb{A}_K)_\bullet^{f'_e})$. Again using [21, Lemma 5.3(5)] we have $Z(\mathbb{A}_K)_\bullet^{f_e} = Z(\mathbb{A}_K)_\bullet^{f'_e}$, so $x \in g(Z(\mathbb{A}_K)_\bullet^{f_e}) \subset X(\mathbb{A}_K)_\bullet^{f}$ which is a contradiction. □

2.2. **The case $G(k^s) = 0$.** It remains to prove Theorem 1.1 in the case that G does not contain a nontrivial étale subscheme. This follows immediately from the next result.

Theorem 2.3. *Let G be a finite abelian group scheme over a global field k such that $G(k^s) = 0$. Let X be a smooth projective variety admitting a torsor $f : Y \to X$ under G. Then for any finite Galois extension L/k, $\left(X(\mathbb{A}_L)_\bullet^f\right)^{\mathrm{Gal}(L/k)} = X(\mathbb{A}_k)_\bullet^f$.*

Lemma 2.4. *Let G be a finite abelian group scheme over field K and let L/K be a Galois extension such that $G(L) = 0$. Then the restriction map $H^1(K, G) \to H^1(L, G)^{\mathrm{Gal}(L/K)}$ is an isomorphism.*

Proof. The inflation-restriction sequence in flat cohomology (see [18, p. 422]) gives an exact sequence

$$H^1(L/K, G(L)) \stackrel{\inf}{\to} H^1(K, G) \stackrel{\mathrm{res}}{\to} H^1(L, G)^{\mathrm{Gal}(L/K)} \to H^2(L/K, G(L))\,.$$

The outer two terms are trivial because $G(L) = 0$ as L/K is separable. Thus, the restriction map is an isomorphism. □

Proof of Theorem 2.3. Noting that $X(\mathbb{A}_L)_\bullet^{\mathrm{Gal}(L/k)} = X(\mathbb{A}_k)_\bullet$, it will be enough to show that we have $X(\mathbb{A}_k)_\bullet^f = X(\mathbb{A}_L)_\bullet^f \cap X(\mathbb{A}_k)_\bullet$. The inclusion $\subseteq$ is clear, so we show the reverse inclusion. We have a commutative diagram

$$\begin{array}{ccc} H^1(k, G) & \hookrightarrow & \prod_v H^1(k_v, G) \\ \big\downarrow & & \big\downarrow \\ H^1(L, G) & \hookrightarrow & \prod_v \prod_{w|v} H^1(L_w, G) \end{array} \tag{1}$$

where the injectivity of the vertical maps come from Lemma 2.4 and the injectivity of the horizontal maps is [8, MainTheorem].

Let $x \in X(\mathbb{A}_L)_\bullet^f \cap X(\mathbb{A}_k)_\bullet$ and consider the image x^*f of x under the map $X(\mathbb{A}_k)_\bullet \to \prod_v H^1(k_v, G)$ induced by f. The image of x^*f in $\prod_v \prod_{w|v} H^1(L_w, G)$ is the image of a unique $\xi \in H^1(L, G)$ under the bottom horizontal map of (1) because $x \in X(\mathbb{A}_L)_\bullet^f$. For any v there is a natural action of $\mathrm{Gal}(L/k)$ on $\prod_{w|v} H^1(L_w, G)$ which is compatible with the action of $\mathrm{Gal}(L/k)$ on $H^1(L, G)$. The image of x^*f in $\prod_{w|v} H^1(L_w, G)$ is fixed by this action, so we conclude that $\xi \in H^1(L, G)^{\mathrm{Gal}(L/k)}$. By Lemma 2.4 we have the ξ is the image of some $\xi' \in H^1(k, G)$. Since the maps are all injective, x^*f must be the image of ξ'. This means $x \in X(\mathbb{A}_k)_\bullet^f$. □

Remark 2.5. In the proof above, the condition $G(L) = 0$ is only used to ensure the existence of a lift of $x^*(f)$ from $\prod_v H^1(k_v, G)$ to $H^1(k, G)$, using that its image in $\prod_v \prod_{w|v} H^1(L_w, G)$ comes from an element in $H^1(L, G)$. The conclusion of Theorem 2.3 therefore holds for any (potentially infinite) group scheme G and any Galois extension L/k such that Diagram (1) is Cartesian.

3. Subvarieties of abelian varieties

Throughout this section $X \subset A$ denotes a smooth closed subvariety of an abelian variety over a global field k.

Definition 3.1. We say that $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ *defines an étale sheaf* if, for any finite Galois extension L/k, we have $\left(X(\mathbb{A}_L)_\bullet^{\mathrm{Br}}\right)^{\mathrm{Gal}(L/k)} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$, and similarly for the other sets appearing in (†). This condition is equivalent to the assignment $L \mapsto X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ defining a sheaf of sets on $\mathrm{Spec}(k)_{\acute{e}t}$.

3.1. **The finite abelian descent set.** Following [21, p. 352] we define $\widehat{\mathrm{Sel}}(A/k) = \varprojlim_n \mathrm{Sel}^n(A/k)$. This fits into the Cassels-Tate dual exact sequence [15, Proposition 4.3], which reads

$$0 \to \widehat{\mathrm{Sel}}(A/k) \to A(\mathbb{A}_k)_\bullet \xrightarrow{\phi} H^1(k, A^\vee)^* \,.$$

The map ϕ is induced by the sum of the local Tate pairings $\langle\,,\,\rangle_{k_v} : A(k_v) \times H^1(k_v, A^\vee) \to \mathbb{Q}/\mathbb{Z}$. This sequence identifies $\widehat{\mathrm{Sel}}(A/k)$ with a subset of $A(\mathbb{A}_k)_\bullet$.

Theorem 3.2. *The sets* $\widehat{\mathrm{Sel}}(A/k)$ *and* $X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(A/k)$ *define étale sheaves.*

Proof. Let L/k be any finite Galois extension. It suffices to prove the result for $\widehat{\mathrm{Sel}}(A/k)$, since

$$\left(X(\mathbb{A}_L)_\bullet \cap \widehat{\mathrm{Sel}}(A/L)\right)^{\mathrm{Gal}(L/k)} = X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(A/L)^{\mathrm{Gal}(L/k)} \,.$$

Suppose $x \in \widehat{\mathrm{Sel}}(A/L)^{\mathrm{Gal}(L/k)} = \widehat{\mathrm{Sel}}(A/L) \cap A(\mathbb{A}_k)_\bullet$ and let $d = [L : k]$. We first claim that $dx \in \widehat{\mathrm{Sel}}(A/k) \subset A(\mathbb{A}_k)_\bullet$. For this we use the Cassels-Tate dual exact sequence. For any place v, passing to an extension L^v/k_v of degree d_v multiplies the local Tate pairing by d_v, i.e., we have $\langle x, \mathrm{res}_{L^v/k_v}(\alpha)\rangle_{L^v} = d_v\langle x, \alpha\rangle_{k_v}$. From this we deduce that for any finite extension K/k we have $[K : k]\phi(x) = \phi_K(x) \circ \mathrm{res}_{K/k} \in H^1(k, A^\vee)^*$. The assumption that $x \in \widehat{\mathrm{Sel}}(A/L)^{\mathrm{Gal}(L/k)}$ together with exactness of the sequence gives $\phi_L(x) = 0$, so $\phi(dx) = d\phi(x) = \phi_L(x) \circ \mathrm{res}_{L/k} = 0$, showing that $dx \in \widehat{\mathrm{Sel}}(A/k)$.

Since $dx \in \widehat{\mathrm{Sel}}(A/k) = \varprojlim_n \mathrm{Sel}^n(A/k)$, there is a compatible system of geometrically connected torsors $f_n : Y_n \to A$ under $A[n]$ containing lifts $y_n \in Y_n(\mathbb{A}_k)$ of dx. Here compatible means that for each m, n, we have a torsor structure $Y_{mn} \to Y_n$ under $A[m]$ sending y_{mn} to y_n. The trivial torsor $[d] : A \to A$ contains a lift of dx by hypothesis, so f_d must be a twist of this trivial covering by an element $\xi \in Ш^1(k, A[d])$. For each $n \geq 1$, let $g_{nd} : Z_{nd} \to A$ be the twist of f_{nd} by the image of $-\xi$ under the map $Ш^1(k, A[d]) \to Ш^1(k, A[nd])$ induced by the inclusion $A[d] \hookrightarrow A[nd]$. Then $g_d = [d] : A \to A$ and g_{dn} factors as $g_{dn} = [d] \circ h_n$ for some torsor $h_n : Z_{nd} \to A$ under $A[n]$. Since ξ is locally trivial and the f_{nd} contain lifts of dx, we have we have $dx^* g_{nd} = dx^* f_{nd} = 0$, for all n. It follows that the family h_n determines an element in $\widehat{\mathrm{Sel}}(A/k)$, and consequently an adelic point $x' \in A(\mathbb{A}_k)_\bullet$. By construction $dx' = dx$, so $x - x' \in A[d](\mathbb{A}_k)_\bullet$ is an adelic point contained in a finite subscheme of A.

By assumption $x' - x$ lies in $\widehat{\mathrm{Sel}}(A/L)$. Using [21, Proposition 3.6] in the number field case and [15, Proposition 5.3] in the function field case (Note that the additional hypothesis on A there can be dropped thanks to work of Rössler; see [6, Proposition 3.1]) we conclude that $x' - x \in A(L)$. Then $x' - x \in A(L) \cap A(\mathbb{A}_k)_\bullet = A(k)$. So $x \in x' + A(k) \subset \widehat{\mathrm{Sel}}(A/k)$ as required. □

Proof of Theorem 1.2. As noted on [21, p. 373] the image of $\widehat{\mathrm{Sel}}(A/k)$ in $A(\mathbb{A}_k)_\bullet$ is equal to $A(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. This follows from the fact that the Néron-Severi group of A is torsion free, and so the geometrically abelian fundamental group of A is isomorphic to its Tate module. The same argument works to show $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}}(A/k)$ for any $X \subset A$ with torsion free Néron-Severi group. So Theorem 1.2 follows from Theorem 3.2. □

As an amusing corollary of Theorem 1.2 we have the following.

Corollary 3.3. *Suppose $X \subset A$ is defined over a number field and satisfies the following:*

(1) $\overline{X(k)}$ *defines an étale sheaf,*
(2) *the Néron-Severi group of X is torsion free, and*
(3) $\mathrm{Br}(X_{k^s}) = 0$.

If $\overline{X(L)} = X(\mathbb{A}_L)_\bullet^{\mathrm{Br}}$ for all finite separable extensions L/k, then $\overline{X(L)} = X(\mathbb{A}_L)_\bullet^{\text{f-ab}}$ for all finite separable extensions L/k. In other words, if the Brauer-Manin obstruction is the only obstruction to weak approximation for such X, then the a priori weaker obstruction coming from finite abelian descent is already sufficient to capture this obstruction.

Proof. Assumption (1) is that $\mathcal{F}(L) = \overline{X(L)} = X(\mathbb{A}_L)_\bullet^{\mathrm{Br}}$ is an étale sheaf, and $\mathcal{G}(L) = X(\mathbb{A}_L)_\bullet^{\text{f-ab}}$ is an étale sheaf by Theorem 1.2. The inclusion $X(\mathbb{A}_L)_\bullet^{\mathrm{Br}} \subset X(\mathbb{A}_L)_\bullet^{\text{f-ab}}$ shows that $\mathcal{F}$ is a subsheaf of $\mathcal{G}$. It is enough to show that this inclusion is an isomorphism étale locally, for then $\mathcal{F} = \mathcal{G}$ by [9, §II Proposition 1.1], in which case we have $\overline{X(L)} = \mathcal{F}(L) = \mathcal{G}(L) = X(\mathbb{A}_L)_\bullet^{\text{f-ab}}$ for any separable extension L/k.

By condition (3) we have $\mathrm{Br}(X) = \mathrm{Br}_1(X) := \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{k^s}))$. Condition (2) gives that $\mathrm{NS}(X_{k^s})$ is a finitely generated free $\mathbb{Z}$-module, so $H^1(K, \mathrm{NS}(X_{k^s})) = 0$ for all sufficiently small neighbourhoods K/k. For such K/k we have $X(\mathbb{A}_K)_\bullet^{\mathrm{Br}} = X(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ as in [21, Corollary 7.8], showing that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic étale locally. □

Remark 3.4. Condition (1) of Corollary 3.3 is satisfied if $X \subset A$ is coset free, since in this case $X(L)$ is finite for all L/k. Using [4, Lemma 3.1], Condition (3) can be replaced by the weaker condition that $\mathrm{Br}(X_{k^s})$ is finite, or the even weaker condition

(3') For every sufficiently small neighbourhood K/k of $\mathrm{Spec}(k)_{\text{ét}}$, there is a separable extension L/K such that
  (a) $[L : K]\,\mathrm{Br}(X_K) \subset \mathrm{Br}_1(X_K)$, and
  (b) $\mathrm{res}_{L/K} : \mathrm{Br}(X_K)/\mathrm{Br}_1(X_K) \to \mathrm{Br}(X_L)/\mathrm{Br}_1(X_L)$ is surjective.

3.2. **Sheafiness of the other terms in (†).**

Lemma 3.5. *The sets $\overline{A(k)}$ and $X(\mathbb{A}_k)_\bullet \cap \overline{A(k)}$ define étale sheaves.*

Proof. The topological closure of $A(L)$ and the profinite completion $\widehat{A(L)}$ are isomorphic as $\mathrm{Gal}(L/k)$-modules by [15, Theorem E]. Since $A(L)$ is finitely generated we also have an isomorphism of $\mathrm{Gal}(L/k)$-modules $\widehat{A(L)} \simeq A(L) \otimes \widehat{\mathbb{Z}}$. Then $(A(L) \otimes \widehat{\mathbb{Z}})^{\mathrm{Gal}(L/k)} = A(L)^{\mathrm{Gal}(L/k)} \otimes \widehat{\mathbb{Z}} = A(k) \otimes \widehat{\mathbb{Z}}$, the latter being identified with $\overline{A(k)}$ by [15, Theorem E]. Thus $\overline{A(k)}$ defines an étale sheaf. The statement about $X(\mathbb{A}_k)_\bullet \cap \overline{A(k)}$ follows instantly since

$$\left(X(\mathbb{A}_L)_\bullet \cap \overline{A(L)}\right)^{\mathrm{Gal}(L/k)} = X(\mathbb{A}_L)_\bullet^{\mathrm{Gal}(L/k)} \cap \overline{A(L)}^{\mathrm{Gal}(L/k)} = X(\mathbb{A}_k)_\bullet \cap \overline{A(k)}.$$

□

Lemma 3.6. *If k is a number field or the maximal divisible subgroup of Ш(A/k) is trivial, then $A(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ defines an étale sheaf.*

Proof. In the number field case, [3, Theorem 1] implies $A(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = A(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, so this follows from Theorem 3.2. If the divisible subgroup of Ш(A/k) is trivial then $\overline{A(k)} = A(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = A(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ (See [15, Remark 4.5]), so this follows from either Lemma 3.5 or Theorem 3.2. □

Remark 3.7. When A/k is an isotrivial abelian variety, we have that Ш(A/k) is finite. For A/k constant, this is shown in [11], and as mentioned in Remark 6.27 of [12] it is an easy extension to the isotrivial case.

Lemma 3.8. *If $X \subset A$ is a curve embedded in its Jacobian, then $\overline{X(k)}$ defines an étale sheaf.*

Proof. If X has genus 1, then $X = A$ and this follows from Lemma 3.5. So suppose X has genus ≥ 2. If k is a number field or X is a nonisotrivial, then $X(k)$ is finite. Then $\overline{X(k)} = X(k)$ which clearly defines an étale sheaf. We may therefore suppose X is an isotrivial curve of genus ≥ 2. This case is proved in Lemma 4.9 of the following section. □

Theorem 3.9. *Suppose $X \subset A$ is a curve embedded in its Jacobian by an Albanese map (i.e., a map sending a point P to the class of $P - D$ for a fixed k-rational divisor $D \in \mathrm{Div}(X)$ of degree 1). Then all of the sets in (†) define étale sheaves.*

Proof. First note that $X(\mathbb{A}_k)_\bullet \cap A(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet \cap A(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ (See [21, Corollary 7.3]). So all of these define étale sheaves by Theorem 3.2. Moreover, $X(\mathbb{A}_k)_\bullet \cap \overline{A(k)}$ and $\overline{X(k)}$ define étale sheaves by Lemma 3.5 and Lemma 3.8. □

Remark 3.10. Suppose C is a curve of genus ≥ 1 that does not have a k-rational divisor of degree 1, so that it is not embedded in its Jacobian. Then C embeds canonically in Pic^1_C which is a nontrivial torsor under $A = \mathrm{Jac}(C)$. If $C(\mathbb{A}_k)^{\mathrm{Br}}_\bullet \neq \emptyset$, then Pic^1_C is a nontrivial divisible element in $Ш(A/k)$ by [19, Proposition 3.3.5 and Theorem 6.1.2]. By Remark 3.7 this cannot occur if C is isotrivial, so for isotrivial curves not embedded in their Jacobian, all of the sets in (†) are trivial.

3.3. **Counterexamples among general varieties.** We now show that, in general, none of the sets in the top row of (†) define étale sheaves, if we consider varieties that do not admit an embedding into an abelian variety.

Proposition 3.11. *Let Y/k be a smooth projective variety over a number field such that* $\mathrm{Pic}(\overline{Y})$ *is torsion free,* $\mathrm{Br}(\overline{Y})$ *is finite, and* $\mathrm{Br}(Y) \to \mathrm{Br}(\overline{Y})^{\mathrm{Gal}(k)}$ *is surjective.*

1. *If $Y(\mathbb{A}_k)_\bullet \neq Y(\mathbb{A}_k)^{\mathrm{Br}}_\bullet$, then there exists a finite Galois extension L/k such that*
$$\left(Y(\mathbb{A}_L)^{\mathrm{Br}}_\bullet\right)^{\mathrm{Gal}(L/k)} \neq Y(\mathbb{A}_k)^{\mathrm{Br}}_\bullet .$$
2. *If $Y(k) = \emptyset$, $Y(\mathbb{A}_k)_\bullet \neq \emptyset$ and $\overline{Y(K)} = Y(\mathbb{A}_K)^{\mathrm{Br}}_\bullet$ for all finite Galois extensions K/k, then there exists a finite Galois extension L/k such that $\overline{Y(L)}^{\mathrm{Gal}(L/k)} \neq \emptyset = \overline{Y(k)}$.*

Proof. The assumptions in the first sentence imply that $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$ is finite, say of order d. By [4, Lemma 3.3] there are infinitely many Galois extensions of degree divisible by d such that the restriction map $\mathrm{Res}_{L/k} : \mathrm{Br}(Y)/\mathrm{Br}_0(Y) \to \mathrm{Br}(Y_L)/\mathrm{Br}_0(Y_L)$ is surjective. For any such extension, [4, Lemma 3.1(2)] gives that $Y(\mathbb{A}_k)_\bullet \subset Y(\mathbb{A}_L)^{\mathrm{Br}}_\bullet$, so every element $Y(\mathbb{A}_k)_\bullet$ lies in the $\mathrm{Gal}(L/k)$-invariant subset of $Y(\mathbb{A}_L)^{\mathrm{Br}}_\bullet$. The claims in (1) and (2) follow immediately. □

Remark 3.12. Examples satisfying the conditions of Proposition 3.11(1) can be found among del Pezzo surfaces and Châtelet surfaces. Châtelet surfaces that are counterexamples to the Hasse principle satisfy the conditions in Proposition 3.11(2) by [2, Theorem B].

Proposition 3.13. *Let Y/k be a smooth projective Enriques surface over a global field k of characteristic not equal to 2. Then the assignment $L \mapsto Y(\mathbb{A}_L)^{\text{f-ab}}_\bullet$ does not define an étale sheaf.*

Proof. Let $f : Z \to Y$ be a $K3$ cover of the Enriques surface. Then Z is étale simply connected, and so $Y(\mathbb{A}_L)^f_\bullet = Y(\mathbb{A}_L)^{\text{ét}}_\bullet = Y(\mathbb{A}_L)^{\text{f-ab}}_\bullet$. The result follows by Theorem 2.1. □

4. Isotrivial curves

In this section we use results of the previous sections to generalize the main results of [5] to the case of isotrivial curves.

Fix a finite field $\mathbb{F}$ of characteristic p. Let D be a smooth projective geometrically connected curve over $\mathbb{F}$, and let $k = \mathbb{F}(D)$ denote the function field of D. Throughout this section C will denote a smooth projective and geometrically connected curve over k of genus $g = g(C) \geq 2$ which we assume to be isotrivial. We also assume that C has a k-rational divisor z of degree 1 and use this to define an embedding of C into its Jacobian $J = \mathrm{Jac}(C)$ by the rule $x \mapsto [x - z]$. This assumption is justified by Remark 3.10.

There exists a finite extension L/k with corresponding extension of constant fields $\mathbb{F}_L/\mathbb{F}$ and a curve $C_0/\mathbb{F}_L$ such that $C \times_k L \simeq C_0 \times_{\mathbb{F}_L} L$. We note that for any such L/k we also have that $J \times_k L \simeq J_0 \times_{\mathbb{F}_L} L$, where $J_0 = \mathrm{Jac}(C_0)$ is an abelian variety defined over $\mathbb{F}_L$. One can take L/k to be separable because the moduli space of curves with sufficiently large level structure is a fine moduli space. Moreover, replacing L by its Galois closure we obtain a Galois extension of k trivialising C.

Remark 4.1. There are isotrivial varieties that are not separably isotrivial, meaning that they only become constant after a non-separable field extension, e.g. singular genus-changing curves in the sense of [22]. It is possible that there exist smooth examples in higher dimension but we do not know of any.

The set of places of k is in bijection with the set D^1 of closed points of D. For $v \in D^1$ we denote the residue field of the completion k_v by $\mathbb{F}_v$. Note that since the valued field k_v has equicharacteristic p, $\mathbb{F}_v \subseteq k_v$. We define $\mathbb{A}_{k,\mathbb{F}} := \prod_{v\in D^1} \mathbb{F}_v$, which is an $\mathbb{F}$-subalgebra of the usual adele ring over $\mathbb{A}_k$. If L/k is a finite extension with constant field extension $\mathbb{F}_L/\mathbb{F}$, then there exists a smooth projective curve $D'/\mathbb{F}_L$ with $L = \mathbb{F}_L(D')$, so we may define $\mathbb{A}_{L,\mathbb{F}_L}$ similarly.

4.1. **Locally constant adelic points.** We recall the definition of locally constant adelic points as in [5, Section 2.2] (where they were called reduced adelic points) before extending this definition to isotrivial varieties. Suppose that X/k is a constant variety so that $X = X_0 \times_{\mathbb{F}} k$. Since $D/\mathbb{F}$ is geometrically connected, we have natural equalities of sets

$$\mathrm{Hom}_{/k}(\mathrm{Spec}(\mathbb{A}_k), X) = \mathrm{Hom}_{/\mathbb{F}}(\mathrm{Spec}(\mathbb{A}_k), X) = \mathrm{Hom}_{/\mathbb{F}}(\mathrm{Spec}(\mathbb{A}_k), X_0)\,.$$

Definition 4.2. For a constant variety $X = X_0 \times_{\mathbb{F}} k$ over k we define the *locally constant adelic points* to be the set

$$X(\mathbb{A}_{k,\mathbb{F}}) := \mathrm{Hom}_{/\mathbb{F}}(\mathrm{Spec}(\mathbb{A}_{k,\mathbb{F}}), X_0)\,.$$

Note that $\mathbb{A}_{k,\mathbb{F}}$ is an $\mathbb{F}$-subalgebra of $\mathbb{A}_k$, so we have an inclusion

$$X(\mathbb{A}_{k,\mathbb{F}}) \subseteq \mathrm{Hom}_{/\mathbb{F}}(\mathrm{Spec}(\mathbb{A}_k), X_0) = \mathrm{Hom}_{/k}(\mathrm{Spec}(\mathbb{A}_k), X) = X(\mathbb{A}_k)\,.$$

Concretely, $X(\mathbb{A}_{k,\mathbb{F}}) = \prod_{v\in D^1} X_0(\mathbb{F}_v)$.

Definition 4.3. Suppose X/k is an isotrivial variety and L/k is a Galois extension with corresponding residue extension $\mathbb{F}_L/\mathbb{F}$ such that X_L/L is a constant variety. Let $X_0/\mathbb{F}_L$ be the corresponding constant variety and let $\phi : X_L \to X_0 \times_{\mathbb{F}_L} L$ be an isomorphism. Define the *locally constant adelic points* of X/k to be the set

$$X(\mathbb{A}_{k,\mathbb{F}}) := X(\mathbb{A}_L)^{\mathrm{Gal}(L/k)} \cap \phi^{-1}((X_0 \times_{\mathbb{F}_L} L)(\mathbb{A}_{L,\mathbb{F}_L})) \subset X(\mathbb{A}_L)^{\mathrm{Gal}(L/k)} = X(\mathbb{A}_k)\,.$$

The following two lemmas show that this definition does not depend on our choice of trivialising extension L and isomorphism ϕ.

Lemma 4.4. *Let X/k be an isotrivial variety, and let $\phi, \phi' : X_L \to X_0 \times_{\mathbb{F}} L$ be two isomorphisms of L varieties. Then $\phi^{-1}((X_0 \times_{\mathbb{F}} L)(\mathbb{A}_{L,\mathbb{F}})) = \phi'^{-1}((X_0 \times_{\mathbb{F}} L)(\mathbb{A}_{L,\mathbb{F}}))$. In particular, the set $X(\mathbb{A}_{k,\mathbb{F}})$ does not depend on our choice of trivialising isomorphism ϕ.*

Proof. Note that $\phi' \circ \phi^{-1}$ gives us an automorphism of $X_0 \times_{\mathbb{F}} L$, so the result is equivalent to saying that the set $X_0 \times_{\mathbb{F}} L(\mathbb{A}_{L,\mathbb{F}})$ is stable under automorphisms of $X_0 \times_{\mathbb{F}} (\mathbb{A}_{L,\mathbb{F}})$. This is immediate by the definition of locally constant adelic points for constant varieties. □

Lemma 4.5. *Let X/k be an isotrivial variety. Then the definition of $X(\mathbb{A}_{k,\mathbb{F}})$ does not depend on our choice of trivialising extension.*

Proof. Let L, L' be two trivialising extensions for X/k, and consider the extension $K := LL'$. Since X_L is constant, we have that $X_L(\mathbb{A}_{L,\mathbb{F}_L}) = X_K(\mathbb{A}_{K,\mathbb{F}_K}) \cap X_L(\mathbb{A}_L)$, and similarly for $X_{L'}(\mathbb{A}_{L',\mathbb{F}_{L'}})$. Therefore

$$X_L(\mathbb{A}_{L,\mathbb{F}_L}) \cap X(\mathbb{A}_k) = X_K(\mathbb{A}_{K,\mathbb{F}_K}) \cap X(\mathbb{A}_k) = X_{L'}(\mathbb{A}_{L',\mathbb{F}_{L'}}) \cap X(\mathbb{A}_k)$$

as required. □

Remark 4.6. Locally constant adelic points are defined precisely so that they define an étale sheaf in the sense of Definition 3.1, and their use is justified by the property they are shown to satisfy in Theorem 4.11.

4.2. **The Frobenius map on isotrivial varieties.**

Lemma 4.7. *Suppose X/k is an isotrivial variety and L/k is a Galois extension such that there is an isomorphism $X_L \simeq X_0 \times_{\mathbb{F}_L} L$ for some $X_0/\mathbb{F}_L$ where $\mathbb{F}_L/\mathbb{F}$ is the residue extension corresponding to L/k. Let $m := [\mathbb{F}_L : \mathbb{F}]$. The relative Frobenius morphism $F_{X_0/\mathbb{F}_L} : X_0 \to X_0$ induces a morphism induces a morphism $F_{X_L/L} : X_L \to X_L$ that is compatible with descent data, and so yields a morphism $X \to X$, which we call $F^m_{X/k}$.*

Example 4.8. Consider an isotrivial curve $C : ty^2 = f(x)$ over $\mathbb{F}_p(t)$ with $f(x) \in \mathbb{F}_p[x]$ for an odd prime p. The relative Frobenius for X/k is the morphism $F : C \to C^{(p)}$ given on coordinates by raising to the p-th power, where $C^{(p)}/\mathbb{F}$ is the curve given by $t^p y^2 = f(x)$. Since p is odd, we have $C^{(p)} \simeq C$ by the map $(x, y) \mapsto (x, t^{\frac{p-1}{2}} y)$. The Galois extension $L = \mathbb{F}_p(t^{1/2})/\mathbb{F}_p(t)$ trivializes C. The morphism $F^1_{X/k}$ constructed in the lemma is the composition of F and the isomorphism $C^{(p)} \simeq C$.

Proof of Lemma 4.7. Let q be the cardinality of $\mathbb{F}_L$. On any open subset U of X_L the morphism $F_{X/L}$ is given by

$$F_{X/L} : \mathcal{O}_{X_L}(U) \to \mathcal{O}_{X_L}(U)$$
$$x \mapsto x^q\,.$$

From this it is clear that $F_{X_L/L}$ commutes with the descent data on X_L/L. □

4.3. **Proof of Lemma 3.8 in the isotrivial case.** The following lemma completes the proof of Lemma 3.8 which stated that $\overline{C(k)}$ defines an étale sheaf.

Lemma 4.9. *Let L/k be a Galois extension. Then $\overline{C(L)}^{\mathrm{Gal}(L/k)} = \overline{C(k)}$.*

Proof. We claim that we can assume without loss of generality that L/k trivialises C/k, i.e., L/k is such that $C \times_k L \simeq C_0 \times_{\mathbb{F}_L} L$ for some curve $C_0/\mathbb{F}_L$. Let K/k be a Galois extension such that $L \subseteq K$ and K/k trivialises C/k. Then $\overline{C(K)}^{\mathrm{Gal}(K/k)} = (\overline{C(K)}^{(\mathrm{Gal}(K/L))})^{\mathrm{Gal}(L/k)}$, so showing the result for extensions that trivialise C/k implies the result for all Galois extensions.

Let D_L be the smooth projective curve with $L = \mathbb{F}_L(D_L)$. Let $F = F_{C_0/\mathbb{F}_L}$ be the relative Frobenius morphism. By a theorem of de Franchis the set $C(L)/F$ is finite. Define a set of distinct elements $\phi_1, \ldots, \phi_m \in C(L) = \mathrm{Mor}_{\mathbb{F}_L}(D_L, C_0)$ representing all morphisms from D_L to C_0 up to Frobenius twisting. Concretely this means that for any $\phi \in C(L)$, there exists integers $a, b \geq 0$ and an $i \in \{1, \ldots, m\}$ such that $F^a\phi = F^b\phi_i$. We can assume that the ϕ_i are distinct, separable and the set $\{\phi_1, \ldots, \phi_m\}$ is closed under the action of $\mathrm{Gal}(L/k)$. The map $C(L)/F \to \mathrm{Map}(D^1_L, C^1_0)$ is injective by [20, Proposition 2.3] and there are only finitely many ϕ_i, so we can find a finite set S of places $v \in D^1_L$ such that the elements $r_S(\phi_i) := (r_v(\phi_i))_{v \in S} \in \prod_{v \in S} C_0(\mathbb{F}_v)$ are distinct for $i \neq j$, where $r_v : C_0(L_v) \to C_0(\mathbb{F}_v)$ is the reduction map. Enlarging S if needed, we can assume that $r_S(F^a\phi_i) \neq r_S(F^b\phi_j)$ for any $a, b \geq 0$ and $i \neq j$. Note that $r_v(F^a\phi_i) = F^a(r_v(\phi_i))$, where on the right F acts via $\mathrm{Gal}(\mathbb{F}_L)$ on $C_0(\mathbb{F}_v)$.

Let ρ_v be metrics inducing the v-adic topology on $C(L_v)$ and let ρ be a product metric of these inducing the adelic topology on $C(\mathbb{A}_L)$. The reduction maps give rise to a continuous retraction $r = (r_v) : C(\mathbb{A}_L) \to C(\mathbb{A}_{L,\mathbb{F}_L})$. By the discussion above, the set of distances $\{\rho(r(F^a\phi_i), r(F^b\phi_j)) \mid a, b \geq 0 \text{ and } i \neq j\}$ is bounded away from 0.

Suppose $P_n \in C(L)$ converge to $P \in \overline{C(L)} \cap C(\mathbb{A}_k)$. For any $\sigma \in \mathrm{Gal}(L/k)$ the sequence $\sigma(P_n)$ converges to $\sigma(P) = P$ in $C(\mathbb{A}_L)$, since the induced map $\sigma : C(\mathbb{A}_L) \to C(\mathbb{A}_L)$ is continuous. Thus the sequence of real numbers $\rho(P_n, \sigma(P_n))$ converges to 0. For all $n \geq 1$ we have integers $a_n, b_n \geq 0$ and $i_n \in \{1, \ldots, m\}$ such that $F^{a_n}P_n = F^{b_n}\phi_{i_n}$. Since $F : C(\mathbb{A}_L) \to C(\mathbb{A}_L)$ is continuous and commutes with the action of σ, we find that the sequence $\rho(F^{b_n}\phi_{i_n}, F^{b_n}\sigma(\phi_{i_n}))$ converges to 0. Since the reduction map is continuous this implies that $\rho(r(F^{b_n}\phi_{i_n}), r(F^{b_n}\sigma(\phi_{i_n})))$ converge to 0. These distances are bounded away from zero when $\phi_{i_n} \neq \sigma(\phi_{i_n})$. So for all large enough n

we must have $\sigma(\phi_{i_n}) = \phi_{i_n}$. Since the action of σ commutes with F we find that, for all large enough n, we have $F^{a_n}\sigma(P_n) = F^{a_n}P_n$ and so $P_n = \sigma(P_n)$ as well. Thus the P_n are eventually in $C(L)^{\mathrm{Gal}(L/k)} = C(k)$. Hence $P \in \overline{C(k)}$. □

4.4. **Frobenius descent.** Let L/k be the minimal Galois extension trivializing the isotrivial curve C/k. Then L/k also trivializes $J = \mathrm{Jac}(C)$ and Lemma 4.7 gives an isogeny $F^m_{J/k} : J \to J$ where $m = [\mathbb{F}_L : \mathbb{F}]$.

Lemma 4.10. *The locally constant adelic points of the Jacobian satisfy* $J(\mathbb{A}_{k,\mathbb{F}}) \subset F^m_{J/k}(J(\mathbb{A}_k))$.

Proof. This is clear for constant varieties and Frobenius commutes with the Galois action. □

For any $n \geq 1$, the n-fold composition of $F^m_{J/k}$ is an isogeny $\phi_n = (F^m_{J/k})^{\circ n} : J \to J$ whose kernel is a finite connected abelian group scheme. The pullback of ϕ_n along the embedding $C \to J$ is a torsor $F^n : Y \to C$ under $\ker(\phi_n)$. We define $C(\mathbb{A}_k)^{F^\infty} = \bigcap_{n\geq 1} C(\mathbb{A}_k)^{F^n}$.

The following generalizes [5, Theorem 1.2] to the case of isotrivial curves.

Theorem 4.11. *Only global and locally constant adelic points survive infinite Frobenius descent, i.e.,* $C(\mathbb{A}_k)^{F^\infty} = C(k) \cup C(\mathbb{A}_{k,\mathbb{F}})$.

Proof. For a finite separable extension L/k with corresponding residue extension $\mathbb{F}_L/\mathbb{F}$ define $\mathcal{F}(L) := C(L) \cup C(\mathbb{A}_{L,\mathbb{F}_L})$ and $\mathcal{G}(L) := C(\mathbb{A}_L)^{\ker(F^\infty)}$. Then $\mathcal{F}$ and $\mathcal{G}$ are sheaves of sets on $\mathrm{Spec}(k)_{\text{ét}}$, the latter by Theorem 2.3. By Lemma 4.10, all reduced adelic points survive Frobenius descent, so $\mathcal{F}$ is a subsheaf of $\mathcal{G}$. For any L/k trivializing C, we may apply [5, Theorem 1.2] in order to obtain $\mathcal{F}(L) = \mathcal{G}(L)$. We conclude that $\mathcal{F}(k) = \mathcal{G}(k)$ as in the proof of Theorem 4.13. □

4.5. **The Mordell-Weil Sieve.**

Definition 4.12. We define the *Mordell–Weil Sieve set* for an isotrivial curve embedded in its Jacobian to be the set

$$C^{\text{MW-sieve}} := C(\mathbb{A}_{k,\mathbb{F}}) \cap \overline{J(k)}\,,$$

with this intersection taking place in $J(\mathbb{A}_k)$.

When C/k is a constant curve this agrees with the definition of $C^{\text{MW-sieve}}$ in [5] so the following result generalizes [5, Theorem 1.1] to isotrivial curves.

Theorem 4.13. *For C/k a smooth, projective, isotrivial curve embedded in its Jacobian, we have an equality* $C(\mathbb{A}_k)^{\mathrm{Br}} = C(k) \cup C^{\text{MW-sieve}}$.

Proof. The assignments $L \mapsto \mathcal{G}(L) := C(\mathbb{A}_L)^{\mathrm{Br}}$ and $L \mapsto \mathcal{F}(L) := C(\mathbb{A}_{L,\mathbb{F}'}) \cap \overline{J(L)}$ define étale sheaves by Theorem 3.9 and Lemma 3.5. Moreover, $\mathcal{F}(L) \subset \mathcal{G}(L)$ for all finite extensions L/k since $C(\mathbb{A}_L)^{\mathrm{Br}} \subset J(\mathbb{A}_L)^{\mathrm{Br}} = \overline{J(L)}$, where the final equality is given in the proof of Lemma 3.6, noting that $Ш(A/L)$ is finite by Remark 3.7. For any L/k trivializing C we have $\mathcal{F}(L) = \mathcal{G}(L)$ by [5, Theorem 1.1], so we proceed as in the proof of Theorem 4.13 to obtain $\mathcal{G}(k) = \mathcal{F}(k)$ as required. □

Remark 4.14. If C is an isotrivial curve that does not admit an embedding in its Jacobian, Remark 3.10 shows that $C(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. For C/k not isotrivial and of genus ≥ 2, [6, Theorem 1.1] gives that $C(\mathbb{A}_k)^{\mathrm{Br}} = C(k)$. The above result is a step towards a characterisation of $C(\mathbb{A}_k)^{\mathrm{Br}}$ for curves in the remaining cases.

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