# QUARTIC DEL PEZZO SURFACES WITHOUT QUADRATIC POINTS

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Abstract. Previous work of the authors showed that every quartic del Pezzo surface over a number field has index dividing 2, and asked whether such surfaces always have a closed point of degree 2. We resolve this by constructing infinitely many quartic del Pezzo surfaces over $\mathbb{Q}$ without degree 2 points. These are the first examples of smooth intersections of two quadrics with index strictly less than the minimal degree of a closed point.

The index of a variety over a field is the GCD of the degrees of its closed points. Theorems of Springer, Amer and Brumer [Spr56, Ame76, Bru78] together imply that a smooth intersection of two quadrics has index 1 if and only if it has a closed point of degree 1.

The purpose of this note is to prove that this does not extend to the case of larger index.

**Theorem 1.** *There are infinitely many smooth intersections of two quadrics over $\mathbb{Q}$ that have index 2, but have no closed points of degree 2.*

In earlier work [CV23], the authors proved that smooth intersections of two quadrics of dimension at least 2 over number fields always have index dividing 2 and, moreover, that degree 2 points always exist in dimension at least 2 over local fields, and in dimension at least 3 over global fields (assuming Schinzel's hypothesis in the number field case).[1] Colliot-Thélène subsequently proved the result over number fields unconditionally [CT24, Théorème 1.5]. In dimension 1, it is possible to have index 1, 2 or 4 over local and global fields, but index and minimal degree coincide by the Riemann-Roch theorem. Thus, any smooth intersection of quadrics over a global field with index strictly less than the minimal degree of a closed point must be dimension 2, in which case it is a quartic del Pezzo surface.

The proof of Theorem 1 is constructive. In Section 1, we introduce the family $X_d$ of quartic del Pezzo surfaces of interest, prove some preliminary results about the local isometry type of quadric threefolds that contain them, and use [CV23] to show that, under some assumptions on $d$, the quartic del Pezzo $X_d$ is not contained in any $\mathbb{Q}$-rational quadric threefold that contains a $\mathbb{Q}$-line. This almost immediately yields Theorem 1 (see Section 1.4 for details).

In personal communication with the authors, Colliot-Thélène sketched a proof, developed jointly with Kollár, that any quartic del Pezzo surface with index 2 has a point of degree 2, 6 or 14. In Section 2.1, we observe that the results in [CV23] imply that, assuming Schinzel's hypothesis, quartic del Pezzo surfaces over number fields always have degree 6 points.

We close by giving an indication of how these examples arise. In Section 2.2, we explain how the 2- and 3-adic constraints in our examples ensure we are in the exceptional case where [CV23, Theorem 1.2(5)] does not give degree 2 points. In Section 2.3, we explain how a result of Flynn [Fly09] enabled us to produce equations of surfaces with our desired properties that have small coefficients and so are amenable to exploration.

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[1]Over general fields there are examples in dimension 2 with index 4; see [CV23, Section 7C].

## 1. A family of quartic del Pezzo surfaces

For a nonzero integer $d$, let $X_d \subset \mathbb{P}^4_{\mathbb{Q}}$ be the surface defined by the vanishing of the quadratic forms

$$\begin{aligned} Q_0 &:= d(u_0^2 - 3u_1^2) + 6u_2^2 + 6u_2u_3 - 6u_3u_4\,, \\ Q_\infty &:= -2du_0u_1 + u_2^2 - 2u_2u_4 - 3u_3^2 - u_4^2\,. \end{aligned} \tag{1}$$

Corollary 7 below shows that for suitable choices of $d$ (e.g., $d = -7$) the surface $X_d$ has no closed points of degree 2.

1.1. **The pencil of quadrics.** The forms in (1) define a pencil of quadrics in $\mathbb{P}^4 \times \mathbb{P}^1$ whose fiber over $t := (a : b) \in \mathbb{P}^1$ is the quadric 3-fold $V(bQ_0 + aQ_\infty) \subset \mathbb{P}^4$; the quadrics in this pencil are exactly the quadric threefolds that contain $X_d$. For a point $(a : 1) \in \mathbb{P}^1$ we will write $Q_a = Q_0 + aQ_\infty$. Let $M_0, M_\infty$ be the symmetric matrices corresponding to $Q_0, Q_\infty$ and let

$$\Phi(T) := \det(M_0 + TM_\infty) = -6d^2(T^2 + 3)(T^3 + 3T^2 + 3T - 9)\,.$$

Let $\mathcal{S} \subset \mathbb{P}^1$ be the subscheme defined by the vanishing of $\Phi(T)$; the singular quadrics in the pencil all lie above $\mathcal{S}$. Let $\mathcal{T}$ and $\mathcal{T}'$ be the degree 2 and degree 3 subschemes defined by the irreducible factors of $\Phi(T)$, and let $\zeta$ denote a root of the cubic factor of $\Phi$. The discriminant of $Q_{\mathcal{T}}$ is $[d] \in \mathbf{k}(\mathcal{T})^\times/\mathbf{k}(\mathcal{T})^{\times 2}$ and the discriminant of $Q_{\mathcal{T}'}$ is $[\zeta] \in \mathbf{k}(\mathcal{T}')^\times/\mathbf{k}(\mathcal{T}')^{\times 2}$.

1.2. **Lines on the quadrics over local fields.**

**Lemma 2.** *Let $Q$ be a rank 5 quadratic form over a field $k$ of characteristic not equal to 2 and assume that $Q$ is an orthogonal sum of a hyperbolic plane and a rank 3 form. The quadric 3-fold $V(Q) \subset \mathbb{P}^4_k$ contains a $k$-rational line if and only if the conic in $\mathbb{P}^2_k$ defined by the rank 3 form contains a $k$-rational point.*

*Proof.* We can assume $Q = u_0u_1 + q(u_2, u_3, u_4)$. The hyperplane tangent to $V(Q)$ at the $k$-point $P = (0 : 1 : 0 : 0 : 0)$ is $V(u_0)$. The intersection $V(Q, u_0)$ is a cone over the conic $V(q) \subset \mathbb{P}^2_k$. Hence, if the conic contains a $k$-point, then $V(Q)$ contains a $k$-rational line. On the other hand, if $V(Q)$ contains a $k$-rational line, then $Q$ has Witt index 2 and so contains two hyperbolic planes. By Witt cancellation, the rank 3 form must also contain a hyperbolic plane. Equivalently, this means that the conic has a $k$-point. □

**Lemma 3.** *Let $v$ be a place of $\mathbb{Q}$. The quadric $V(Q_\infty)$ contains a $\mathbb{Q}_v$-rational line if and only if* $\mathrm{inv}_v(2, 3) = 0$ *which is if and only if $v \notin \{2, 3\}$.*

*Proof.* The form $Q_\infty$ is (over $\mathbb{Q}$) the direct sum of a hyperbolic plane and the rank 3 form $u_2^2 - 2u_2u_4 - 3u_3^2 - u_4^2$. The latter diagonalizes as $(u_2 - u_4)^2 - 3u_3^2 - 2(u_4)^2$ and, hence, corresponds to the quaternion algebra $(2, 3) \in \mathrm{Br}(\mathbb{Q})$ ramified at $v = 2, 3$. Now apply Lemma 2. □

**Lemma 4.** *For all $b \in \mathbb{Z}_3$, the quadric $V(bQ_0 + Q_\infty) \subset \mathbb{P}^4_{\mathbb{Q}_3}$ contains no $\mathbb{Q}_3$-rational lines.*

*Proof.* First note that $\mathcal{S}(\mathbb{Q}_3) = \{(\zeta : 1)\}$ and $v_3(\zeta) > 0$, so $(1 : b) \notin \mathcal{S}$. This means that $bQ_0 + Q_\infty$ has rank 5. It the orthogonal sum of the rank 2 form $d(bu_0^2 - 2u_0u_1 - 3bu_1^2)$ and a rank 3 form which diagonalizes as

$$-(u_4 + u_2 + 3bu_3)^2 + 2(1 + 3b)\left(u_2 + \frac{6b}{2 + 6b}u_3\right)^2 - 3\left(1 + 3b^2 \cdot \frac{1 - 3b}{1 + 3b}\right)u_3^2\,.$$

The condition on $b$ implies that the rank 2 form is a hyperbolic plane over $\mathbb{Q}_3$, hence we may apply Lemma 2 to $bQ_0 + Q_\infty$ over $\mathbb{Q}_3$. For any $b \in \mathbb{Z}_3$, $1 + 3b^2 \cdot \frac{1-3b}{1+3b}$ and $1 + 3b$ are squares in $\mathbb{Z}_3$. Thus, the conic corresponds to the quaternion algebra $(2, -3)$ and so has no $\mathbb{Q}_3$-points. □

**Remark 5.** For $v|6$, one can find quadrics in the pencil containing a $\mathbb{Q}_v$-line, as is implied by the local case in [CV23, Theorem 1.2]. As the existence of these quadrics is not needed for the proof of Theorem 1, we omit the details.

### 1.3. Lines on the quadrics over $\mathbb{Q}$.

**Theorem 6.** *Suppose the integer $d$ satisfies the following:*

*(i)* $d \equiv 5 \bmod 12$,

*(ii) For all primes $v$ dividing $d$, we have that $\zeta$ is a square in $\mathbb{Q}_v \otimes \mathbb{Q}(\zeta)$.*

*Then no quadric in $\mathbb{P}^4_{\mathbb{Q}}$ containing $X_d$ contains a line defined over $\mathbb{Q}$.*

*Proof.* Any quadric in $\mathbb{P}^4_{\mathbb{Q}}$ containing $X_d$ must lie in the pencil spanned by $Q_0$ and $Q_\infty$. Let $\pi \colon \mathcal{G} \to \mathbb{P}^1$ be the scheme such that $\mathcal{G}_t := \pi^{-1}(t)$ parameterizes the lines on the quadric 3-fold $V(Q_t) \subset \mathbb{P}^4$. Then $\mathcal{G}$ is smooth over $\mathbb{Q}$ and we must show that $\mathcal{G}(\mathbb{Q}) = \emptyset$. We will do this by showing that there is a Brauer-Manin obstruction to rational points on $\mathcal{G}$.

By [CV23, Proposition 5.1] we have that $\mathrm{Br}(\mathcal{G})$ is generated modulo constant algebras by

$$\beta := \pi^*(d, T^2 + 3) = \pi^*\mathrm{Cor}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(d, T - \sqrt{-3})\,, \tag{2}$$

where $(d, T^2 + 3)$ denotes the quaternion algebra in $\mathrm{Br}(\mathbf{k}(\mathbb{P}^1)) = \mathrm{Br}(\mathbb{Q}(T))$. By [CV23, Proposition 5.1] and Lemma 3, $\beta$ may also be expressed as

$$\beta = \pi^*\,\mathrm{Cor}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta, T - \zeta) + [\mathcal{G}_\infty] = \pi^*\left(\mathrm{Cor}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta, T - \zeta) + (2, 3)\right)\,. \tag{3}$$

Let $S = \{2, 3, 5, \infty\} \cup \{v : v \mid d\}$. The discriminant of $\Phi(T)$ is an $S$-unit, so the equations defining the pencil give a smooth model of $\mathcal{G}$ over $\mathbb{Z}_S$. Then, for $v \notin S$, the evaluation map $\mathrm{inv}_v \circ \beta \colon \mathcal{G}(\mathbb{Q}_v) \to \mathbb{Q}/\mathbb{Z}$ is constant by the results of [CTS13]. By evaluating at points of $\mathcal{G}_\infty$ (which exist by Lemma 3), we deduce that $\mathrm{inv}_v \circ \beta$ is identically 0 for $v \notin S$.

Now consider $v \in S$. If $d$ is a square in $\mathbb{Q}(\sqrt{-3}) \otimes_{\mathbb{Q}} \mathbb{Q}_v$, then (2) gives that $\mathrm{inv}_v \circ \beta = 0$. This holds for $v \in \{2, \infty\}$, since $d \equiv 1 \bmod 4$ by (i). If $\zeta$ is a square in $\mathbb{Q}(\zeta) \otimes_{\mathbb{Q}} \mathbb{Q}_v$, then (3) gives that $\mathrm{inv}_v \circ \beta = \mathrm{inv}_v(2, 3)$. This holds for $v = 5$ by an easy computation and for $v \mid d$ by Condition (ii). Moreover, $d$ is prime to 6 by (i), so $\mathrm{inv}_v \circ \beta = \mathrm{inv}_v(2, 3) = 0$ for all $v \mid 5d$. Finally, consider $v = 3$. By Lemma 4, if $P \in \mathcal{G}(\mathbb{Q}_3)$, then $\pi(P) = (a : 1)$ for some $a \in 3\mathbb{Z}_3$. Since $d \equiv 2 \bmod 3$ by (i), we then have $\mathrm{inv}_3(\beta(P)) = \mathrm{inv}_3(d, a^2 + 3) = \frac{1}{2}$.

Thus, for any $(P_v) \in \prod_v \mathcal{G}(\mathbb{Q}_v) = \mathcal{G}(\mathbb{A}_{\mathbb{Q}})$, we have $\sum_v \mathrm{inv}_v(\beta(P_v)) = \mathrm{inv}_3(P_v) = \frac{1}{2}$. This means there is a Brauer-Manin obstruction to the existence of $\mathbb{Q}$-points on $\mathcal{G}$. Indeed, any $\mathbb{Q}$-point $P \in \mathcal{G}(\mathbb{Q}) \subset \mathcal{G}(\mathbb{A}_{\mathbb{Q}})$ would have $\beta(P) \in \mathrm{Br}(\mathbb{Q})$ so $\sum_v \mathrm{inv}_v(\beta(P)) = 0 \neq \frac{1}{2}$. □

### 1.4. The Proof of Theorem 1.

**Corollary 7.** *If $d$ satisfies the conditions of Theorem 6, then the surface $X_d$ has no closed points of degree dividing 2.*

*Proof.* If $X_d$ contains a degree 2 point, then the line in $\mathbb{P}^4_{\mathbb{Q}}$ between that point and its conjugate is defined over $\mathbb{Q}$ and lies on some quadric in the pencil. But by Theorem 6, no quadric in the pencil contains a $\mathbb{Q}$-line, so $X_d$ has no degree 2 points. A similar argument shows there are no degree 1 points; see [CV23, Proposition 4.1] for further details. Alternatively, one can simply note that $X_d(\mathbb{Q}_3) = \emptyset$ to deduce that $X_d(\mathbb{Q}) = \emptyset$. □

**Lemma 8.** *Let $d, d'$ be two nonzero integers. If $dd' \notin \mathbb{Q}(\sqrt{-3})^{\times 2}$, then $X_d \not\simeq X_{d'}$.*

*Proof.* The anticanonical embedding of a quartic del Pezzo surface $X$ presents $X$ as an intersection of quadrics in $\mathbb{P}^4$. Thus, $X_d \simeq X_{d'}$ if and only if there is a change of coordinates of $\mathbb{P}^4$ that gives the isomorphism. A linear automorphism of $\mathbb{P}^4$ will preserve the field of definition, the rank and the discriminant of any quadric hypersurface. Note that each of $X_d$ and $X_{d'}$ is contained in exactly two rank 4 quadrics defined over $\mathbb{Q}(\sqrt{-3})$ with discriminant $d\mathbb{Q}(\sqrt{-3})^{\times 2}$ and $d'\mathbb{Q}(\sqrt{-3})^{\times 2}$ respectively. Thus, if $dd' \notin \mathbb{Q}(\sqrt{-3})^{\times 2}$, then $X_d \not\simeq X_{d'}$. □

*Proof of Theorem 1.* Corollary 7 and Lemma 8 imply that each squarefree $d$ that satisfies the conditions of Theorem 6 gives a distinct quartic del Pezzo surface with no points of degree 1 or 2. Thus, by the Springer and Amer-Brumer theorem, the index of $X_d$ is greater than 1. Hence, by [CV23, Theorem 1.1], $X_d$ must have index equal to 2. It remains to show that there are infinitely many squarefree integers $d$ that satisfy the conditions of Theorem 6.

Consider the Galois closure $L$ of $\mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{\zeta})$. By the Chebotarev Density Theorem, there is an infinite set of primes $\mathcal{P}$ that split completely in $L$. We claim that for all $p \in \mathcal{P}$, we have that $p \equiv 1 \bmod 12$ and that $\zeta$ is a square in $\mathbb{Q}_p \otimes \mathbb{Q}(\zeta)$. This claim together with the fact that $\zeta$ is a square in $\mathbb{Q}_7 \otimes \mathbb{Q}(\zeta)$ then implies that for any finite subset $S \subset \mathcal{P}$, the integer $d = -7\prod_{p\in S} p$ will satisfy the conditions of Theorem 6.

Now let us prove the claim. If $p$ splits completely in $L$, then it splits completely in $\mathbb{Q}(\sqrt{-1})$, in $\mathbb{Q}(\sqrt{-3})$ and in $\mathbb{Q}(\sqrt{\zeta})$. Thus, $-1, -3 \in \mathbb{Q}_p^\times$, which, by quadratic reciprocity, implies that $p \equiv 1 \bmod 4$ and $p \equiv 1 \bmod 3$. Further, since $p$ splits completely in the Galois closure of $\mathbb{Q}(\sqrt{\zeta})$, the cubic equation $T^3 + 3T^2 + 3T - 9$ has three roots over $\mathbb{Q}_p$, and, moreover, each root is a square. □

## 2. Complements

2.1. **Degree 6 points on quartic del Pezzo surfaces.** Every quartic del Pezzo surface over a number field has a closed point of degree equal to 2 mod 4 by [CV23, Theorem 1.1]. The proof constructs an adelic 0-cycle of degree 1 on the corresponding variety $\mathcal{G}$ that is orthogonal to $\mathrm{Br}(\mathcal{G})$. A closer examination of the proof shows that one can also construct an *effective* adelic 0-cycle of degree 3 that is orthogonal to $\mathrm{Br}(\mathcal{G})$. This allows us to deduce the following.

**Theorem 9.** *Let $X$ be a smooth quartic del Pezzo surface over a number field. Assume Schinzel's hypothesis. Then $X$ has a closed point of degree 6.*

*Proof.* It will be enough to show that there is a cubic extension $K/k$ such that, for the variety $\mathcal{G}/k$ parameterizing lines on quadrics in the pencil of quadrics containing $X$, we have $\mathcal{G}(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$. Indeed, we may then apply an unpublished result of Serre that, assuming Schinzel's hypothesis, the Brauer-Manin obstruction is the only obstruction to the Hasse principle for pencils of Severi-Brauer varieties, thus giving $\mathcal{G}(K) \neq \emptyset$ and consequently a closed point of degree 2 on $X_K$.

If $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, then by [CV23, Lemma 3.2] we have $\mathcal{G}(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$ for any extension $K/k$. So we may assume that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. Hence, we are in the exceptional case that is not covered by [CV23, Corollary 6.1], which is discussed in [CV23, Remark 6.2 and Section

7A]. We have that $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ is cyclic, generated by an element $\beta_{\mathcal{T}} \in \mathrm{Br}(\mathcal{G})$ of order 2 corresponding to an irreducible degree 2 subscheme $\mathcal{T}$ of the locus of singular quadrics in the pencil containing $X$. Moreover, the discussion in [CV23, Section 7A] gives a place $v$ of $k$ (in the set $R'_{\mathcal{T}}$ described there) such that $v$ does not split in $\mathbf{k}(\mathcal{T})$ and the evaluation map $\beta_{\mathcal{T}} : \mathcal{G}(\mathbf{k}(\mathcal{T}_{k_v})) \to \mathrm{Br}(\mathbf{k}(\mathcal{T}_{k_v}))$ is nonconstant. Let $K/k$ be any cubic extension such that $K \otimes k_v \simeq \mathbf{k}(\mathcal{T}_{k_v}) \times k_v$. Then $K$ has place $w \mid v$ such that $K_w \simeq \mathbf{k}(\mathcal{T}_{k_v})$ and for this place the evaluation map $\mathrm{inv}_w \circ \beta_{\mathcal{T}} : \mathcal{G}(K_w) \to \mathbb{Q}/\mathbb{Z}$ is nonconstant. It follows that $\mathcal{G}(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$. ☐

**Remark 10.** The proof above gives explicit cubic extensions $K/k$ such that $\mathcal{G}(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$. For $X = X_d$ with $d$ as in Theorem 6, it gives that $\mathcal{G}(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$ for the cubic extension $K = \mathbb{Q}(\zeta)$. For specific values of $d$ it is possible to check directly that $\mathcal{G}(\mathbb{Q}(\zeta)) \neq \emptyset$ by searching for quadrics in the pencil which contain $K$-rational lines (which can be checked by computing the Hasse invariant of the corresponding quadratic form). For example, when $d = -7$ we find that $\mathcal{G}_t(\mathbb{Q}(\zeta)) \neq \emptyset$ for the point $t = (5\zeta + 3 : 7) \in \mathbb{P}^1(\mathbb{Q}(\zeta))$.

2.2. **Comparison with [CV23, Theorem 1.2].** We will now show that the surfaces $X_d$ in the theorem do not satisfy the hypotheses of [CV23, Theorem 1.2(5)] (so the results of that paper imply only that $X_d$ has a 0-cycle of degree 2, not necessarily a quadratic point). Specifically, $X = V(Q_{\sqrt{-3}}) \cap V(Q_{-\sqrt{-3}})$ for the singular quadrics $V(Q_{\pm\sqrt{-3}}) \subset \mathbb{P}^4_{\mathbb{Q}(\sqrt{-3})}$ and there are an odd number of places of $\mathbb{Q}(\sqrt{-3})$ for which these fail to have smooth local points.

**Lemma 11.** *Assume that $d \equiv 5 \bmod 12$. Then, the singular quadric $V(Q_{\sqrt{-3}}) \subset \mathbb{P}^4_{\mathbb{Q}(\sqrt{-3})}$ has smooth points over all completions of $\mathbb{Q}(\sqrt{-3})$ except at the unique prime of $\mathbb{Q}(\sqrt{-3})$ above 2 (where it does not have smooth points).*

*Proof.* The field $K := \mathbb{Q}(\sqrt{-3})$ is totally imaginary, so it suffices to consider completions at nonarchimedean places $v$. The quadric $V(Q_{\sqrt{-3}})$ has smooth points if and only if the rank 4 quadratic form $Q_{\sqrt{-3}}$ is isotropic. Over a local field there is a unique anisotropic form of rank 4, and it has square discriminant. We rewrite $Q_{\sqrt{-3}}$ as

$$d(u_0 - \sqrt{-3}u_1)^2 - \sqrt{-3}\left(u_4 - \sqrt{-3}u_3 + u_2\right)^2 + 2\sqrt{-3}\left(\sqrt{-3}u_3 - u_2\right)^2 - 2(\sqrt{-3}u_2)^2,$$

and see that the discriminant of a smooth hyperplane section is $dK^{\times 2}$. Thus, if $d \notin K_v^{\times 2}$, the quadric $V(Q_{\sqrt{-3}})$ has smooth $K_v$-points. In particular, since $d \equiv 2 \bmod 3$, $V(Q_{\sqrt{-3}})$ has smooth $K_v$-points for the unique place $v|3$.

Let $v$ be such that $d \in K_v^{\times 2}$ and let $L$ be the étale $K_v$-algebra $K_v[z]/(z^2 - \sqrt{-3})$. Then for a suitable choice of linear forms $\ell_0, \dots \ell_3$, we may write $Q_{\sqrt{-3}}$ as

$$\mathrm{Norm}_{L/K_v}(\ell_0 - z\ell_1) = 2\,\mathrm{Norm}_{L/K_v}(\ell_2 - z\ell_3).$$

In particular, for $v$ such that $d \in K_v^{\times 2}$, $V(Q_{\sqrt{-3}})$ has a smooth $K_v$ point if and only if $2 \in \mathrm{Norm}(L/K_v)$ or, equivalently, if and only if $\mathrm{inv}_v(2, \sqrt{-3})$ is trivial. We compute

$$\mathrm{inv}_v(2, \sqrt{-3}) = \begin{cases} 0 & v \nmid 6, \\ \frac{1}{2} & v \mid 3. \end{cases}$$

By Hilbert reciprocity, we deduce that $\mathrm{inv}_w(2, \sqrt{-3}) = \frac{1}{2}$ for the unique prime $w$ above 2. ☐

**Remark 12.** In the notation of [CV23, Definitions 5.7 and 5.15] the proof of the lemma shows that $C_{\mathcal{T}} := \mathrm{Cor}_{\mathbf{k}(\mathcal{T})/\mathbb{Q}}(\mathrm{Clif}(Q_{\mathcal{T}}))$ is equal to the class in $\mathrm{Br}(\mathbb{Q})$ of the quaternion algebra ramified at 2 and 3 and that $R_{\mathfrak{J}} = \{2\}$. In particular, $R_{\mathfrak{J}}$ has odd cardinality and we are in the exceptional case considered in [CV23, Remark 6.2 and Section 7A] where the results of that paper are not able to show $\mathcal{G}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$.

**Remark 13.** For any rank 4 quadric $V(Q) \subset \mathbb{P}^4_L$ over a quadratic extension $L/k$ of a global field $k$ such that $V(Q)$ is not defined over $k$ and has smooth points over all but an odd number of completions of $L$, [CV23, Corollary 6.3] gives that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \mathcal{G}(\mathbb{A}_k)$ for the variety $\mathcal{G}/k$ corresponding to pencil over $k$ containing $V(Q)$ and its Galois conjugate. To get that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ one must also have that the local evaluation maps are all constant. This is not the case for all surfaces $X_d$ in our family. For example, if $\mathcal{G}/\mathbb{Q}$ corresponds to the surface with $d = -79$, then it satisfies Lemma 11 because $-79 \equiv 5 \bmod 12$, but one can also check that the evaluation map is nonconstant at $v = 79$. We note that the second condition on $d$ in Theorem 6 fails for $d = -79$.

2.3. **Finding surfaces and equations.** We first identified candidate surfaces for the theorem by a computer search which we now describe. All computations were performed using the Magma computational algebra system [BCP97].

Following [Fly09], any degree 4 del Pezzo surface $X$ can be specified by the data $(\mathcal{S}, \varepsilon_{\mathcal{S}})$, where $\mathcal{S} \subset \mathbb{A}^1$ is a reduced degree 5 subscheme and $\varepsilon_{\mathcal{S}} \in \mathbf{k}(\mathcal{S})^\times$ is an element of square norm. Assuming Schinzel's hypothesis, if $X$ has no quadratic points then $\mathcal{S}$ must contain an irreducible degree 2 subscheme [CV23]. So we considered $\mathcal{S}$ defined by polynomials $f(T) = f_2(T)f_3(T)$ with the $f_i(T) \in \mathbb{Z}[x]$ irreducible monic polynomials of degree $i$ of discriminant up to some bound (taken from Magma's number field database). For each $f(T)$ we ran through $\varepsilon_{\mathcal{S}} = (d, \varepsilon)$ with $d$ a nonzero squarefree integer of some bounded size and $\varepsilon \in \mathbb{Q}[T]/\langle f_3(T)\rangle$ one of the finite set of square classes of $S$-units with square norm (for a fixed set $S$ of primes).

For a given pair $(f(T), \varepsilon_{\mathcal{S}})$ equations for the corresponding surface can be computed as follows. Letting $\theta$ denote the image of $T$ in $\mathbb{Q}[T]/\langle f(T)\rangle$, consider the equation

$$(x_1 - \theta x_3)(x_1 - \theta x_3) = \varepsilon_{\mathcal{S}}(u_0 + u_1\theta + u_2\theta^2 + u_3\theta^3 + u_4\theta^4)^2 \,. \tag{4}$$

Multiplying this out and equating coefficients on the basis vectors $1, \theta, \ldots, \theta^4$ of $\mathbb{Q}[T]/\langle f(T)\rangle$ yields five quadratic forms over $\mathbb{Q}$. The coefficients on $\theta^3$ and $\theta^4$ are quadratic forms involving only the $u_i$, and together they define a pencil of quadrics has singular locus and discriminants isomorphic to $(f(T), \delta)$ by [BBFL07, Lemma 17].

In the case we consider, we have that $f(T) = f_2(T)f_3(T)$ is reducible. We found that we obtain quadratic equations in block diagonal form (and with much smaller coefficients) if we express (4) in terms of the basis $(1, 0), (\theta_2, 0), (0, 1), (0, \theta_3), (0, \theta_3^2)$ of $\mathbb{Q}[\theta_2] \times \mathbb{Q}[\theta_3] = \mathbb{Q}[T]/\langle f(T)\rangle$ where $\theta_i$ denotes a root of $f_i(T)$ (as opposed to the power basis.).

Having obtained quadratic forms defining the surface $X = X(f(T), \varepsilon_{\mathcal{S}})$, we proceed to search for $\mathbb{Q}$-points on $\mathcal{G}$ as follows. For a given $t \in \mathbb{P}^1(\mathbb{Q})$, we can check if $\mathcal{G}_t(\mathbb{Q}) \neq \emptyset$ by computing the Hasse invariant of the quadratic form $Q_t$ (which reduces to diagonalizing and computing Hilbert symbols). If no points are found among $t$ of small height, we turn to trying to show that $\mathcal{G}(\mathbb{Q}) = \emptyset$. In this case we check whether the evaluation maps $\mathrm{inv}_v \circ \beta \colon \mathcal{G}(\mathbb{Q}_v) \to \mathbb{Q}/\mathbb{Z}$ appear to be constant for primes $v$ of bad reduction. To do this we find those $t = (a : b)$ among our search space such that $\mathcal{G}_t(\mathbb{Q}_v) \neq \emptyset$ (using the Hasse invariant) and for each such $t$ compute $\mathrm{inv}_v(\beta(\mathcal{G}_t(\mathbb{Q}_v))) = \mathrm{inv}_v(d, f_2(t))$. If all $t$ considered yield the same value, we suspect that the evaluation map is constant and have determined a value in its image.

Following this process we eventually found a surface (the surface $X_d$ with $d = -19$) for which $\mathcal{G}$ had no $\mathbb{Q}$-points on fibers of small height and the evaluation maps all appeared to be constant, with an odd number of them nonzero. Thus, we expected that $\mathcal{G}$ had a Brauer-Manin obstruction to $\mathbb{Q}$-points. Armed with the small height equations (1) for a candidate we were then able to prove (as described in the sections above) that there is in fact a Brauer-Manin obstruction, and that the same proof works for infinitely many other values of $d$.

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