On automorphism groups of toroidal circle planes

Brendan Creutz, Duy Ho, Günter F. Steinke

January 11, 2018

Abstract

Schenkel proved that the automorphism group of a flat Minkowski plane is a Lie group of dimension at most 6 and described planes whose automorphism group has dimension at least 4 or one of whose kernels has dimension 3. We extend these results to the case of toroidal circle planes.

1 Introduction

The classical Minkowski plane is the geometry of plane sections of the standard nondegenerate ruled quadric in real 3-dimensional projective space $\mathbb{P}_3(\mathbb{R})$. It is an example of a flat Minkowski plane, which is an incidence structure defined on the torus satisfying two axioms, namely the Axiom of Joining and the Axiom of Touching. Torodal circle planes are a generalization of flat Minkowski planes in the sense that they are incidence structures on the torus that are only required to satisfy the Axiom of Joining, cf. Subsection 2.1. There are many examples of flat Minkowski planes, cf. [PS01, Section 4.3]. For an example of a proper toroidal circle plane (that is not a flat Minkowski plane), see [Pol98].

The automorphism group of a flat Minkowski plane is well studied. Schenkel [Sch80] showed that this group is a Lie group of dimension at most 6. She also determined flat Minkowski planes whose automorphism groups have dimension at least 4 or one of whose kernels (cf. Subsection 2.4) has dimension 3. Such planes are isomorphic to the classical Minkowski plane, a family constructed by Steinke [Ste85], or a family generalised by Schenkel [Sch80] from the construction by Hartmann [Har81].

In this paper we extend these structural results from flat Minkowski planes to toroidal circle planes. Our first main result is

Theorem 1.1. *With respect to the compact-open topology, the automorphism group of a toroidal circle plane is a Lie group of dimension at most 6.*

Schenkel [Sch80] gave an outline of a proof of Theorem 1.1 in the case of flat Minkowski planes. Her method follows a proof of an analogous result for spherical circle planes by Strambach [Str70] and is based on convergence of automorphisms on so-called triodes. In this approach one has to deal with three different kinds of triodes depending on the number of circles and parallel classes forming their three sides. Furthermore, the upper bound for the dimensions of the automorphism groups initially obtained from this approach is not sharp. An alternative proof in [Ste84] implicitly uses the Axiom of Touching and therefore does not carry over to toroidal circle planes. The proof of Theorem 1.1 we present here avoids these issues.

Based on the results by Schenkel and Theorem 1.1, we also show that a toroidal circle plane will automatically satisfy the Axiom of Touching when its automorphism group is large, in the following sense.

Theorem 1.2. *Let* $\mathbb{T}$ *be a toroidal circle plane with full automorphism group* $\mathrm{Aut}(\mathbb{T})$*. If* $\dim \mathrm{Aut}(\mathbb{T}) \geq 4$ *or one of its kernels is 3-dimensional, then* $\mathbb{T}$ *is a flat Minkowski plane.*

Together with Schenkel's classification (cf. Theorem 2.5), toroidal circle planes satisfying the conditions above are completely determined.

The problem of classifying toroidal circle planes with 3-dimensional automorphism groups remains open. In [Ho17, Section 5.3], it was shown that only certain groups can occur and their possible actions were described. Many families of flat Minkowski planes with 3-dimensional automorphism groups are known, cf. [PS01, Section 4.3], [Ste04], [Ste17], [Ho17, Chapter 6]. There are currently no known examples of proper toroidal circle planes with 3-dimensional automorphism groups. The only known example of a proper toroidal circle plane [Pol98] has a 2-dimensional automorphism group, cf. [Ho17, Section 7.4].

In Section 2, we recall some facts about toroidal circle planes. The proofs of the main theorems are presented in Section 3.

Acknowledgements The authors would like to thank the referee for the careful reading and helpful suggestions. The second author was supported by a UC Doctoral Scholarship.

2 Preliminaries

2.1 Toroidal circle planes, flat Minkowski planes and examples

A *toroidal circle plane* is a geometry $\mathbb{T} = (\mathcal{P}, \mathcal{C}, \mathcal{G}^+, \mathcal{G}^-)$, whose

point set $\mathcal{P}$ is the torus $\mathbb{S}^1 \times \mathbb{S}^1$,

circles (elements of $\mathcal{C}$) are graphs of homeomorphisms of $\mathbb{S}^1$,

$(+)$-parallel classes (elements of $\mathcal{G}^+$) are the verticals $\{x_0\} \times \mathbb{S}^1$,

$(-)$-parallel classes (elements of $\mathcal{G}^-$) are the horizontals $\mathbb{S}^1 \times \{y_0\}$,

where $x_0, y_0 \in \mathbb{S}^1$.

We denote the $(\pm)$-parallel class containing a point p by $[p]_\pm$. When two points p, q are on the same $(\pm)$-parallel class, we say they are $(\pm)$-*parallel* and denote this by $p \parallel_\pm q$. Two points p, q are *parallel* if they are $(+)$-parallel or $(-)$-parallel, and we denote this by $p \parallel q$.

Furthermore, a toroidal circle plane satisfies the following

Axiom of Joining: three pairwise non-parallel points can be joined by a unique circle.

A toroidal circle plane is called a *flat Minkowski plane* if it also satisfies the following

Axiom of Touching: for each circle C and any two nonparallel points p, q with $p \in C$ and $q \notin C$, there is exactly one circle D that contains both points p, q and intersects C only at the point p.

There are various known examples of flat Minkowski planes. For our purpose, we describe two particular families, which play a prominent role in the classification of flat Minkowski planes. We identify $\mathbb{S}^1$ with $\mathbb{R} \cup \{\infty\}$ in the usual way.

Example 2.1 (Half-classical Minkowski plane $\mathcal{M}(f,g)$, cf. [PS01, Subsection 4.3.1])**.** Let f and g be two orientation-preserving homeomorphisms of $\mathbb{S}^1$. Denote $\mathrm{PGL}(2,\mathbb{R})$ by Ξ and $\mathrm{PSL}(2,\mathbb{R})$ by Λ. The circle set $\mathcal{C}(f,g)$ of a *half-classical Minkowski plane* $\mathcal{M}(f,g)$ consists of sets of the form

$$\{(x, \gamma(x)) \mid x \in \mathbb{S}^1\},$$

where $\gamma \in \Lambda \cup g^{-1}(\Xi \backslash \Lambda) f$.

We note that both halves of $\mathcal{M}(f,g)$ consisting of the graphs of orientation-preserving and orientation-reversing homeomorphisms in $\mathcal{C}(f,g)$ are classical (under a suitable coordinate transformation).

Example 2.2 (Generalised Hartmann plane $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$, cf. [PS01, Subsection 4.3.4])**.** For $r, s > 0$, let $f_{r,s}$ be the orientation-preserving *semi-multiplicative homeomorphism* of $\mathbb{S}^1$ defined by

$$f_{r,s}(x) = \begin{cases} x^r & \text{for } x \geq 0, \\ -s|x|^r & \text{for } x < 0, \\ \infty & \text{for } x = \infty. \end{cases}$$

The circle set $\mathcal{C}_{GH}(f,g)$ of a *generalised Hartmann plane* $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$ consists of sets of the form

$$\{(x, sx+t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\},$$

where $s, t \in \mathbb{R}$, $s \neq 0$, sets of the form

$$\left\{ \left(x, \frac{a}{f_{r_1,s_1}(x-b)} + c \right) \,\middle|\, x \in \mathbb{R} \right\} \cup \{(b, \infty), (\infty, c)\},$$

where $a, b, c \in \mathbb{R}$, $a > 0$, and sets of the form

$$\left\{ \left(x, \frac{a}{f_{r_2,s_2}(x-b)} + c \right) \,\middle|\, x \in \mathbb{R} \right\} \cup \{(b, \infty), (\infty, c)\},$$

where $a, b, c \in \mathbb{R}$, $a < 0$.

One can obtain the classical Minkowski plane as the half-classical Minkowski plane $\mathcal{M}(id, id)$, where id denotes the identity map; or alternatively, the generalised Hartmann plane $\mathcal{M}_{GH}(1, 1; 1, 1)$.

2.2 Geometric operations

For a metric space (X, d), let $\mathcal{H}(X)$ be the set of all of its closed subsets. The set $\mathcal{H}(X)$ is a metric space if we equip it with the Hausdorff metric $\mathbf{h}$ induced by the metric d, which is defined by

$$\mathbf{h}(C, D) = \max\{\sup_{y \in D} \inf_{x \in C} d(x, y), \sup_{x \in C} \inf_{y \in D} d(x, y)\},$$

for two closed subsets C and D of X, cf. [PS01, pp. 430, 431].

Considering the point set $\mathcal{P} \cong \mathbb{S}^1 \times \mathbb{S}^1$ as a subset of $\mathbb{R}^3$, we equip $\mathcal{P}$ with the metric $\mathbf{e}$ induced by the Euclidean metric of $\mathbb{R}^3$. The circle set $\mathcal{C}$ is equipped with the Hausdorff metric $\mathbf{h}$ induced by $\mathbf{e}$.

Let $\widetilde{\mathcal{P}^3}$ be the subspace of the product space $\mathcal{P}^3$ consisting of all triples of pairwise nonparallel points. Let $\mathcal{P}^{1,2}$ be the subspace of $\mathcal{H}(\mathcal{P})$ consisting of all 1- or 2-element subsets, that is, $\mathcal{P}^{1,2} := \{\{x, y\} \mid x, y \in \mathcal{P}\}$. Let $\mathcal{C}^{2*}$ be the subspace of the product space $\mathcal{C}^2$ which consists of all pairs of distinct circles that have non-empty intersection. Let $C^{1*} \subset C^{2*}$ be the subspace of pairs of touching circles.

We define five geometric operations on toroidal circle planes as follows.

1. *Joining* $\alpha : \widetilde{\mathcal{P}^3} \to \mathcal{C}$ is defined by $\alpha(x, y, z)$ being the unique circle going through three pairwise nonparallel points x, y, z.
2. *Parallel Intersection* $\pi : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is defined as $\pi : (x, y) \mapsto [x]_+ \cap [y]_-$.
3. *Parallel Projection* $\pi^{+(-)} : \mathcal{P} \times \mathcal{C} \to P$ is defined as $\pi^{+(-)} : (x, C) \mapsto [x]_{+(-)} \cap C$.
4. *Intersection* $\gamma : \mathcal{C}^{2*} \to \mathcal{P}^{1,2}$ is defined as $\gamma(C, D) = C \cap D$.
5. *Touching* $\beta : C^{1*} \to \mathcal{P}$ is defined as $\beta(C, D) = C \cap D$.

Given the metric $\mathbf{e}$ on $\mathcal{P}$, the geometric operations are continuous if and only if $\mathcal{C}$ is equipped with the topology induced by the metric $\mathbf{h}$. For a proof, see [Ho17, Section 3.4].

Toroidal circle planes satisfy a special case of the K4 coherence condition for flat Minkowski planes introduced by Schenkel [Sch80, 2.1].

Lemma 2.3 (K4 coherence condition, special case)**.** *Let $(C_n) \in \mathcal{C}$ be a sequence of circles and $(p_{i,n}) \to p_i, i = 1, 2, 3$ be three converging sequences of points such that $p_{i,n} \in C_n$, $p_1 \parallel_+ p_2$ and $p_3 \notin [p_1]_+$.*

Suppose $(q_n) \to q \subseteq \mathcal{P}$ such that q is not parallel to any p_i. Then $(\pi^-(q_n, C_n)) \to \pi(p_1, q)$ and $(\pi^+(q_n, C_n)) \to \pi(q, p_3)$.

For a proof, see [Sch80, 3.8] or [Ho17, Lemma 4.1.3].

2.3 Derived planes

The *derived plane* $\mathbb{T}_p$ *of* $\mathbb{T}$ *at the point* p is the incidence geometry whose point set is $\mathcal{P} \backslash ([p]_+ \cup [p]_-)$, whose lines are all parallel classes not going through p and all circles of $\mathbb{T}$ going through p. For every point $p \in \mathcal{P}$, the derived plane $\mathbb{T}_p$ is an $\mathbb{R}^2$-plane and even a flat affine plane when $\mathbb{T}$ is a flat Minkowski plane, cf. [PS01, Theorem 4.2.1].

$\mathbb{R}^2$-planes were introduced by Salzmann in the 1950s and have been thoroughly studied, see [Sal67], [Sal+95, Chapter 31] and references therein. Following the notation from [Sal+95, 31.4], we denote the line going through two points a and b by ab. The *closed interval* $[a, b]$ is the intersection of all connected subsets of the line ab that contain a and b; *open intervals* are defined by $(a, b) = [a, b] \backslash \{a, b\}$, *half-open intervals* $[a, b)$ and $(a, b]$ are defined similarly.

One particular result on $\mathbb{R}^2$-planes we will use is the following.

Lemma 2.4 (cf. [Sal+95, Proposition 31.12]). *In $\mathbb{R}^2$-planes, collinearity and the order of (collinear) point triples are preserved under limits, in the following sense:*

(a) If the point sequences $(a_n), (b_n), (c_n)$ have mutually distinct limits a, b, c and if a_n, b_n, c_n are collinear for infinitely many $n \in \mathbb{N}$, then a, b, c are collinear as well.

(b) If, in addition, $b_n \in (a_n, c_n)$ for infinitely many $i \in \mathbb{N}$, then $b \in (a, c)$.

The geometric operations on the derived $\mathbb{R}^2$-plane $\mathbb{T}_p$ at a point p are induced from those on $\mathbb{T}$. On $\mathbb{T}_p$, we coordinatize the point set $\mathbb{R}^2$ in the usual way. It will be convenient to use the maximum metric $\mathbf{d}$ defined as

$$\mathbf{d}(p, q) := \max(|x_1 - x_2|, |y_1 - y_2|),$$

for given two points $p := (x_1, x_2), q := (y_1, y_2)$ in $\mathbb{R}^2$. This metric is equivalent to the restriction of $\mathbf{e}$ to $\mathbb{T}_p$.

2.4 The automorphism group

An *isomorphism between two toroidal circle planes* is a bijection between the point sets that maps circles to circles, and induces a bijection between the circle sets. It can be shown that parallel classes are mapped to parallel classes.

An *automorphism of a toroidal circle plane* $\mathbb{T}$ is an isomorphism from $\mathbb{T}$ to itself. With respect to composition, the set of all automorphisms of a toroidal circle plane is an abstract group. We denote this group by $\mathrm{Aut}(\mathbb{T})$. Every automorphism of a toroidal circle plane is continuous and thus a homeomorphism of the torus, cf. [PS01, Theorem 4.4.1].

Let $C(\mathcal{P})$ be the space of continuous mappings from $\mathcal{P}$ to itself. As each automorphism is an element of $C(\mathcal{P})$, it is natural to equip the automorphism group $\mathrm{Aut}(\mathbb{T})$ with the compact-open topology. Let

$$(A, B) = \{\sigma \in C(\mathcal{P}) \mid \sigma(A) \subset B\},$$

where $A \subset \mathcal{P}$ is compact and $B \subset \mathcal{P}$ is open. The collection of all sets of the form (A, B) is a subbasis for the compact-open topology of $C(\mathcal{P})$.

Since $\mathcal{P}$ is a compact metric space, the compact-open topology is equivalent to the topology of uniform convergence, cf. [Mun75, pp. 283, 286]. Furthermore, the compact-open topology is metrisable and a metric is given by

$$\widetilde{\mathbf{e}}(\sigma, \tau) = \sup\{\mathbf{e}(\sigma(x), \tau(x)) \mid x \in \mathcal{P}\}.$$

With respect to the topology induced from $C(\mathcal{P})$, the group $\mathrm{Aut}(\mathbb{T})$ becomes a topological group (cf. [Are46, Theorem 4] or [Sal+95, 96.6, 96.7]), with a countable basis (cf. [Dug66, Theorem XII.5.2]).

The automorphism group $\mathrm{Aut}(\mathbb{T})$ has two distinguished normal subgroups, the *kernels* T^+ and T^- of the action of $\mathrm{Aut}(\mathbb{T})$ on the set of parallel classes $\mathcal{G}^+$ and $\mathcal{G}^-$, respectively. In other words, the kernel $T^\pm$ consists of all automorphisms of $\mathbb{T}$ that fix every $(\pm)$-parallel class. For convenience, we refer to these two subgroups as the *kernels $T^\pm$ of the plane* $\mathbb{T}$.

For flat Minkowski planes, Schenkel obtained the following results.

Theorem 2.5. *Let $\mathbb{M}$ be a flat Minkowski plane. Then the automorphism group* $\mathrm{Aut}(\mathbb{M})$ *is a Lie group with respect to the compact-open topology of dimension at most 6.*

(a) If $\mathrm{Aut}(\mathbb{M})$ *has dimension at least 5, then* $\mathbb{M}$ *is isomorphic to the classical flat Minkowski plane.*

(b) If $\mathrm{Aut}(\mathbb{M})$ *has dimension 4, then* $\mathbb{M}$ *is isomorphic to one of the following planes.*

(i) A proper half-classical Minkowski plane $\mathcal{M}(f, id)$ *(cf. Example 2.1), where f is a semi-multiplicative homeomorphism of the form* $f_{d,s}$, $(d, s) \neq (1, 1)$*. This plane admits the* 4*-dimensional group of automorphisms*

$$\{(x, y) \mapsto (rx, \delta(y)) \mid r \in \mathbb{R}^+, \delta \in \mathrm{PSL}(2, \mathbb{R})\}.$$

(ii) A nonclassical generalised Hartmann plane $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$ *(cf. Example 2.2), where* $r_1, s_1, r_2, s_2 \in \mathbb{R}^+$, $(r_1, s_1, r_2, s_2) \neq (1, 1, 1, 1)$*. This plane admits the 4-dimensional group of automorphisms*

$$\{(x, y) \mapsto (rx + a, sy + b) \mid a, b, r, s \in \mathbb{R}, r, s > 0\}.$$

(c) If one of the kernels of $\mathbb{M}$ *is 3-dimensional, then* $\mathbb{M}$ *is isomorphic to a plane* $\mathcal{M}(f, id)$*, where f is an orientation-preserving homeomorphism of* $\mathbb{S}^1$*.*

The proof of Theorem 2.5 can be found in [Sch80, Chapter 5], or [PS01, Theorems 4.4.10, 4.4.12, and 4.4.15].

3 Proofs of Main Theorems

Let Σ be the subgroup of $\mathrm{Aut}(\mathbb{T})$ consisting of all automorphisms that leave the sets $\mathcal{G}^{\pm}$ invariant and preserve the orientation of parallel classes. We first have the following.

Lemma 3.1. *In* Σ*, the identity map is the only automorphism that fixes three pairwise non-parallel points.*

For a proof, cf. [PS01, Lemma 4.4.5].

Locally compact transformation groups acting effectively on a 2-dimensional manifold are known to be Lie groups (cf. [PS01, Theorem A2.3.5] or [Sal+95, Theorem 96.31]). Since Σ has finite index in $\mathrm{Aut}(\mathbb{T})$, to prove Theorem 1.1 we will verify that Σ is locally compact. This is done in Lemma 3.4, where we show that Σ is homeomorphic to a closed subset N of the locally compact space $\widetilde{\mathcal{P}^3}$ of dimension 6.

Before introducing the set N, we prove some results that are used in Lemma 3.4.

Lemma 3.2. *Let* $\widetilde{d} := (d_1, d_2, d_3) \in \widetilde{\mathcal{P}^3}$ *be a triple of pairwise nonparallel points. In the derived plane* $\mathbb{T}_{d_1}$*, by means of joining, intersecting, parallel intersecting and parallel projecting, the two points* d_2, d_3 *generate a dense set.*

Proof. Let $d_4 := \pi(d_2, d_3), d_5 := \pi(d_3, d_2)$, where π denotes the operation Parallel Intersection (cf. Subsection 2.2). It is sufficient to show that the set $\mathcal{D}$ generated by the geometric operations is dense in the closed interval $[d_2, d_5]$ (as defined in Subsection 2.3). Suppose for a contradiction that there exists an open interval $(a, b) \subset [d_2, d_5] \backslash \overline{\mathcal{D}}$. Without loss of generality, we may assume $a \in (d_2, b)$, as in Figure 1. We aim to construct a point in $(a, b) \cap \mathcal{D}$ by means of geometric operations.

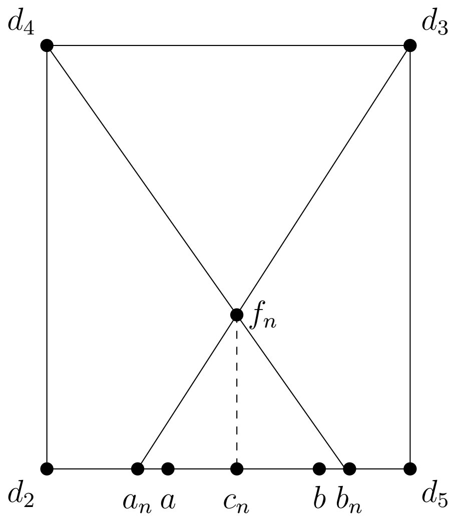

Figure 1

By enlarging the interval if necessary, we may further assume $a, b \in \overline{\mathcal{D}}$. Let (a_n) and (b_n) be sequences convergent to a and b, respectively, such that $a_n \in \mathcal{D} \cap [d_2, a]$, and $b_n \in \mathcal{D} \cap [b, d_5]$. Let $f_n := a_n d_3 \cap d_4 b_n$ and let $c_n := \pi(f_n, d_2)$. Then $c_n \in (a_n, b_n)$. Since the geometric operations are continuous, (f_n) converges to $f := a d_3 \cap d_4 b$, and (c_n) converges to $c := \pi(f, d_5) \in [a, b]$. Since a and d_3 are not parallel, $c \neq a$. Likewise, $c \neq b$, so that $c \in (a, b)$. Then, for all sufficiently large n, the point c_n belongs to (a, b), which contradicts the assumption $(a, b) \cap \mathcal{D} = \emptyset$. Therefore $\mathcal{D}$ is dense in $[d_2, d_5]$. □

Lemma 3.3. *Let $\widetilde{d} := (d_1, d_2, d_3), \widetilde{e} := (e_1, e_2, e_3) \in \widetilde{\mathcal{P}^3}$. As in Lemma 3.2, let $\mathcal{D}$ be the dense set generated by d_2, d_3 in $\mathbb{T}_{d_1}$ and $\mathcal{E}$ be the dense set generated by e_2, e_3 in $\mathbb{T}_{e_1}$. Assume (σ_n) is a sequence of automorphisms in Σ such that $e_i = \lim_n \sigma_n(d_i)$, for $i = 1, 2, 3$. Then*

$$\overline{\sigma} : \mathcal{D} \to \mathcal{E} : x \mapsto \lim_n \sigma_n(x)$$

is well-defined and a homeomorphism between $\mathcal{D}$ and $\mathcal{E}$.

Proof. Let $d_4 := \pi(d_2, d_3), d_5 := \pi(d_3, d_2)$ and $e_i := \lim_n \sigma_n(d_i)$, for $i = 4, 5$. Each point $x \in \mathcal{D}$ can be obtained from the points d_i by finitely many geometric operations. Since we assume (σ_n) converges on d_i, $\lim_n \sigma_n(x)$ exists for $x \in \mathcal{D}$. Hence $\overline{\sigma}$ is well-defined. By construction, $\overline{\sigma}$ is surjective.

1) In the derived plane $\mathbb{T}_{d_1}$, let $\mathcal{R}$ be the closed rectangle formed by the parallel classes of d_2 and d_3. In $\mathbb{T}_{e_1}$, let $\mathcal{S}$ be the closed rectangle formed by the parallel classes of e_2 and e_3, cf. Figure 2. We first prove that if a point $p \in \mathcal{D}$ is in the interior of $\mathcal{R}$, then $q := \overline{\sigma}(p)$ is in the interior of $\mathcal{S}$.

Each automorphism σ_n induces an isomorphism between the derived $\mathbb{R}^2$-planes $\mathbb{T}_{d_1}$ and $\mathbb{T}_{\sigma_n(d_1)}$, cf. [PS01, p. 256]. In particular, $\sigma_n([d_3, d_4]) = [\sigma_n(d_3), \sigma_n(d_4)]$, cf. [Sal67, Theorem 3.5] or [PS01, Theorem 2.4.2]. Since $\pi(p, d_3) \in (d_3, d_4)$, we have $\overline{\sigma}(\pi(p, d_3)) \in [e_3, e_4]$. It follows that $q \in \mathcal{S}$.

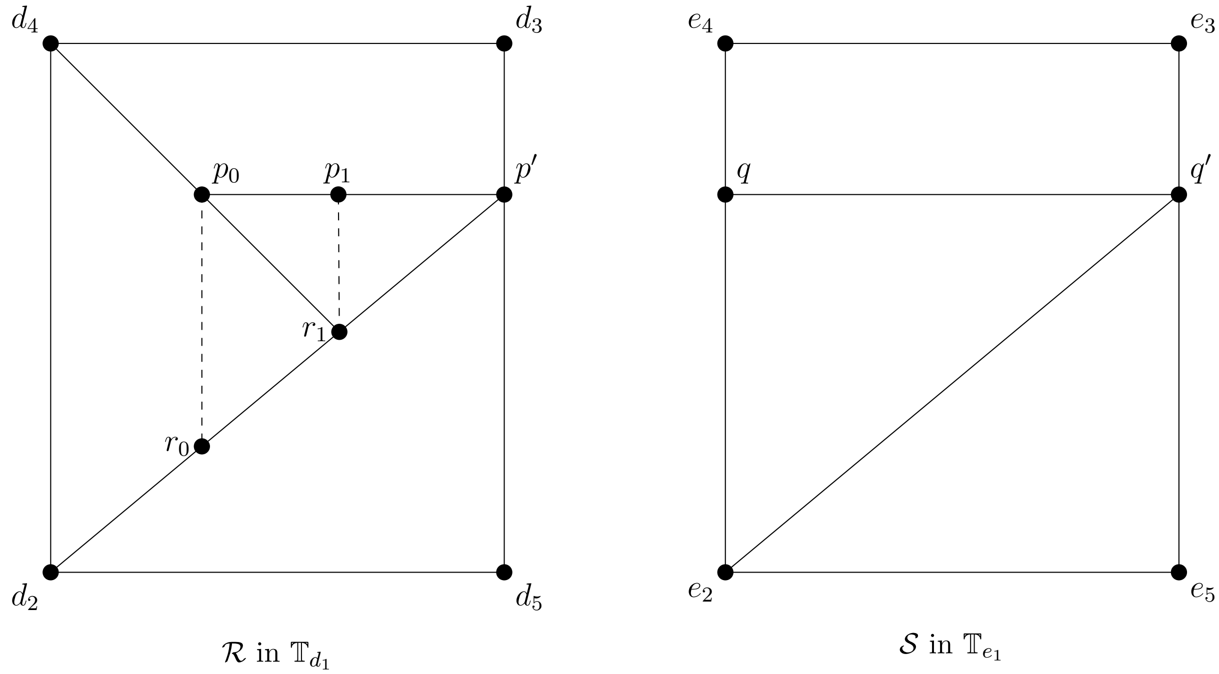

Figure 2

We show that q cannot lie on the boundary of $\mathcal{S}$. Suppose for a contradiction that $q \in [e_2, e_4)$. For inductive purposes, we let $p_0 := p$, $p' := \pi(d_3, p_0)$, and $r_0 := [p_0]_+ \cap d_2p'$. For $n \geq 1$, let $r_n := d_4p_{n-1} \cap d_2p'$, and $p_n := \pi(r_n, p)$. It follows that $r_{n+1} \in (r_n, p')$ and $p_{n+1} \in (p_n, p')$. By applying Lemma 2.3 to the sequence of circles $C_n := \alpha(d_4, p_n, r_{n+1})$ and the constant sequence (d_1) on $\mathbb{T}$, we have both (r_n) and (p_n) convergent to p'.

Let $q' := \overline{\sigma}(p')$. Then $\overline{\sigma}(p_0) = q$ and $\overline{\sigma}(r_0) = [q]_+ \cap e_2q' = e_2$. The inductive construction yields $\overline{\sigma}(p_n) = q$ and $\overline{\sigma}(r_n) = e_2$.

Let $d_6 := d_2d_3 \cap d_4d_5$ and $d_6' := \pi(d_6, p)$. Then there exists an $n \in \mathbb{N}$ such that $p_n \in (d_6', p')$. On the other hand, $\overline{\sigma}(p_n) \in e_2e_4$ implies $\overline{\sigma}(p_n) \notin [\overline{\sigma}(d_6'), q']$, which is a contradiction. Hence, q is in the interior of the rectangle $\mathcal{S}$.

2) By applying the argument in part 1) to arbitrary rectangles in $\mathcal{D}$, we obtain: a) $\overline{\sigma}$ maps nonparallel points to nonparallel points; b) if $(x_n) \in \mathcal{D}$ converges in $\mathbb{T}_{d_1}$, then $(\overline{\sigma}(x_n))$ converges in $\mathbb{T}_{e_1}$. Hence $\overline{\sigma}$ is injective and continuous. Furthermore, if a point $r \in \mathcal{D}$ does not belong to $\mathcal{R}$, then $\overline{\sigma}(r) \notin \mathcal{S}$. In particular, the converse of b) is also true so that $\overline{\sigma}$ is open. This proves the lemma. □

Let N be the subset of $\widetilde{\mathcal{P}^3}$ defined by

$$N := \{(\sigma(d_1), \sigma(d_2), \sigma(d_3)) \mid \sigma \in \Sigma\}.$$

Let $\omega : \Sigma \to N$ be defined by $\omega : \sigma \mapsto (\sigma(d_1), \sigma(d_2), \sigma(d_3))$. From Lemma 3.1 and the definition of N, it follows that ω is a continuous bijection. To prove that ω is a homeomorphism between Σ and N, we rely on the following.

Lemma 3.4. *If $(\sigma_n) \in \Sigma$ is a sequence such that $(\omega(\sigma_n))$ converges in $\widetilde{\mathcal{P}^3}$, then (σ_n) converges in Σ. In particular, ω is a closed map and N is closed in $\widetilde{\mathcal{P}^3}$.*

Proof. 1) We maintain the setup in Lemma 3.3 and its proof. We first claim that (σ_n) converges pointwise on $\mathcal{P}$. Fix $x \in \mathcal{P}$. Let x^* be an accumulation point of $(\sigma_n(x))$. Such a point exists because $\mathcal{P}$ is compact. Passing to subsequences, we may assume $(\sigma_n(x))$ converges to x^*. By changing derived planes if necessary, we can further assume that x^* is a point in $\mathbb{T}_{e_1}$, where $e_1 = \overline{\sigma}(d_1)$ as before.

We finally assume that $x \in (d_2, d_5)$ in $\mathbb{T}_{d_1}$, as other cases can be treated similarly. Let $(y_r) \in \mathcal{D} \cap [d_2, x)$ and $(z_r) \in \mathcal{D} \cap (x, d_5]$ be two sequences convergent to x. Let $\overline{\sigma} : \mathcal{D} \to \mathcal{E} : \xi \mapsto \lim_n \sigma_n(\xi)$ as in Lemma 3.3. Passing to subsequences, we may assume $(\overline{\sigma}(y_r)) \to y^*$ and $(\overline{\sigma}(z_r)) \to z^*$. Here we note that y^* and z^* do not depend on x^* only on x.

In $\mathbb{T}_{e_1}$, by Lemma 2.4, for each fixed r, we have $x^* \in (\overline{\sigma}(y_r), \overline{\sigma}(z_r))$. Letting $r \to \infty$, we get $x^* \in [y^*, z^*]$. Suppose for a contradiction that $y^* \neq z^*$. Since $\mathcal{E}$ is dense and $\overline{\sigma}$ is a bijection between $\mathcal{D}$ and $\mathcal{E}$, there exist two distinct points $u, v \in \mathcal{D}$ such that $\overline{\sigma}(u) \neq \overline{\sigma}(v) \in (y^*, z^*)$. This implies $\overline{\sigma}(u) \in (\overline{\sigma}(y_r), \overline{\sigma}(z_r))$ and thus $u \in (y_r, z_r)$ for all sufficiently large r. The only point in the point set $\mathcal{P}$ satisfying this condition is x, and so $u = x$. The same argument implies that $v = x$, which then yields a contradiction. Therefore $x^* = y^* = z^*$. This proves that the sequence $(\sigma_n(x))$ has a unique accumulation point and thus converges.

2) We now extend the map $\overline{\sigma}$ to the point set $\mathcal{P}$ by letting $\overline{\sigma} : \mathcal{P} \to \mathcal{P} : x \mapsto \lim_n \sigma_n(x)$, which is a well-defined map by part 1). We claim that (σ_n) converges uniformly to $\overline{\sigma}$. It is sufficient to show convergence on the rectangle $\mathcal{R}$ formed by the parallel classes of d_2 and d_3 in the derived plane $\mathbb{T}_{d_1}$.

Fix $\varepsilon > 0$. For every $\xi \in \mathcal{R}$, let $\mathcal{R}_\xi$ be a closed rectangle whose vertices $\xi_1, \xi_2, \xi_3, \xi_4$ belong to $\mathcal{D}$ such that ξ belongs to the interior of $\mathcal{R}_\xi$ and $\max_i \mathbf{d}(\overline{\sigma}(\xi), \overline{\sigma}(\xi_i)) \leq \varepsilon/4$, for $i = 1, \ldots, 4$. This implies $\mathbf{d}(\overline{\sigma}(x), \overline{\sigma}(\xi_i)) \leq \varepsilon/2$ for all $x \in \mathcal{R}_\xi$. By the definition of $\mathbf{d}$, we also have $\mathbf{d}(\overline{\sigma}(x), \overline{\sigma}(y)) \leq \varepsilon/2$ for all $x, y \in \mathcal{R}_\xi$.

The collection of the interiors of all such rectangles $\mathcal{R}_\xi$ is an open cover of $\mathcal{R}$ and thus has a finite subcover $\mathscr{F}$. Let $\mathcal{D}_{\mathscr{F}} \subset \mathcal{D}$ be the set of all vertices of rectangles whose interiors belong to $\mathscr{F}$. Since $\mathcal{D}_{\mathscr{F}}$ is finite and (σ_n) converges pointwise, there exists n_0 such that if $n \geq n_0$ and $\chi \in \mathcal{D}_{\mathscr{F}}$, then $\mathbf{d}(\sigma_n(\chi), \overline{\sigma}(\chi)) \leq \varepsilon/2$.

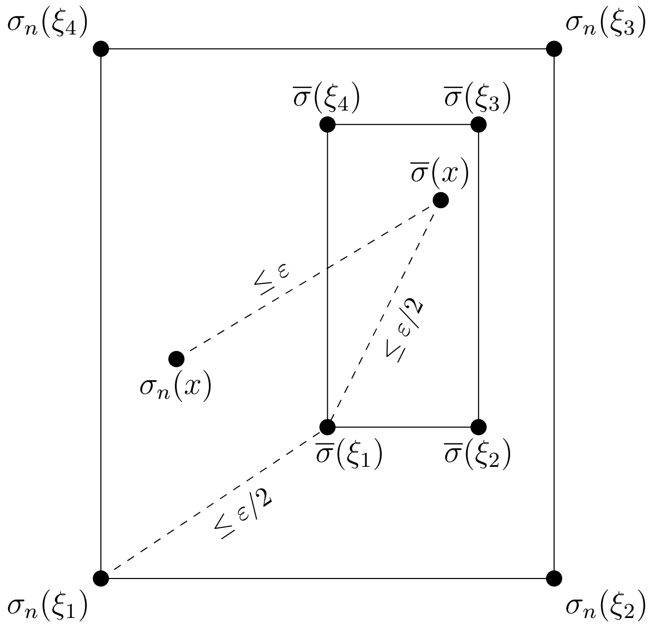

Figure 3

Let $x \in \mathcal{R}$. Then x belongs to the interior of a rectangle $\mathcal{R}_\xi$ in $\mathscr{F}$. Let $\xi_1, \ldots, \xi_4 \in \mathcal{D}_{\mathscr{F}}$ be the vertices of $\mathcal{R}_\xi$. Then $\overline{\sigma}(x)$ belongs to the interior of the rectangle with vertices $\overline{\sigma}(\xi_1), \ldots, \overline{\sigma}(\xi_4)$. In particular, $\max_i \mathbf{d}(\overline{\sigma}(x), \overline{\sigma}(\xi_i)) \leq \varepsilon/2$, see Figure 3.

For $n \geq n_0$, we have $\mathbf{d}(\sigma_n(\xi_i), \overline{\sigma}(\xi_i)) \leq \varepsilon/2$, which implies $\max_i \mathbf{d}(\overline{\sigma}(x), \sigma_n(\xi_i)) \leq \varepsilon$. Since $\sigma_n(x)$ is in the interior of the rectangle with vertices $\sigma_n(\xi_i)$, we have $\mathbf{d}(\sigma_n(x), \overline{\sigma}(x)) \leq 2\varepsilon$. Therefore (σ_n) converges uniformly to $\overline{\sigma}$ on $\mathcal{R}$.

3) As in the proof of Lemma 3.3, it follows that $\overline{\sigma}$ is a bijection and thus a homeomorphism. The continuity of geometric operations ensures that $\overline{\sigma}$ is an automorphism of $\mathbb{T}$. Hence (σ_n) converges to $\overline{\sigma}$ in Σ. □

Proof of Theorem 1.1. We have shown in Lemma 3.4 that $\Sigma \cong N$ and that N is a closed subspace of $\widetilde{\mathcal{P}^3}$. Hence Σ is locally compact and $\dim \Sigma \leq \dim \widetilde{\mathcal{P}^3} = 6$. Since Σ is also a transformation group acting effectively on $\mathcal{P}$, it is a Lie group (cf. [PS01, Theorem A2.3.5] or [Sal+95, Theorem 96.31]). As Σ is a subgroup of index at most 8 of the topological group $\mathrm{Aut}(\mathbb{T})$, it follows that $\mathrm{Aut}(\mathbb{T})$ is a Lie group of the same dimension as Σ. □

Proof of Theorem 1.2. Let Γ be the connected component of $\mathrm{Aut}(\mathbb{T})$ and let $\Delta^\pm$ be the kernel of the action of Γ on $\mathcal{G}^\pm$.

1) Assume $\dim T^+ = 3$. From the proof of [PS01, Theorem 4.4.10], which does not use the Axiom of Touching, it follows that $\mathbb{T}$ is isomorphic to a plane $\mathcal{M}(f, id)$, where f is an orientation-preserving homeomorphism of $\mathbb{S}^1$. Hence $\mathbb{T}$ is a flat Minkowski plane.

2) If $\dim \mathrm{Aut}(\mathbb{T}) \geq 5$, then the same arguments as in the proof of [PS01, Theorem 4.4.12] show that $\mathbb{T}$ is isomorphic to the classical Minkowski plane.

Assume $\dim \mathrm{Aut}(\mathbb{T}) = 4$. If at least one of $\Gamma/\Delta^\pm$ is transitive on $\mathcal{G}^\pm$, then $\mathbb{T}$ is isomorphic to a plane $\mathcal{M}(f, id)$, where f is a semi-multiplicative homeomorphism of the form $f_{d,s}$, $(d, s) \neq (1, 1)$ (cf. [PS01, Theorem 4.4.15]). Otherwise, Γ fixes a point p and acts transitively on $\mathcal{P} \backslash ([p]_+ \cup [p]_-)$. Then Γ induces a 4-dimensional point-transitive group of automorphisms $\overline{\Gamma}$ on the derived plane $\mathbb{T}_p$. By [Sal67, Theorem 4.12], $\mathbb{T}_p$ is isomorphic to either the Desarguesian plane or a half-plane.

Let q be a point in $\mathbb{T}_p$. The stabiliser of q then fixes two lines through q derived from the parallel classes in $\mathbb{T}$. This implies $\mathbb{T}_p$ cannot be a half-plane, since point-stabilisers of half-planes only fix exactly one line through that point. Hence $\mathbb{T}_p$ is a Desarguesian plane. It follows from [PS01, Theorem 4.4.15] that $\mathbb{T}$ is a nonclassical generalised Hartmann plane. □

References

[Are46] Richard Arens. "Topologies for homeomorphism groups". *Amer. J. Math.* 68 (1946), 593–610.

[Dug66] James Dugundji. *Topology.* Boston: Allyn and Bacon Inc., 1966.

[Har81] Erich Hartmann. "Beispiele nicht einbettbarer reeller Minkowski-Ebenen". *Geom. Dedicata* 10. no. 1-4 (1981), 155–159.

[Ho17] Duy Ho. "On the classification of toroidal circle planes". PhD thesis. University of Canterbury, 2017.

[Mun75] James R. Munkres. *Topology: a first course.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975, xvi+413.

[Pol98] Burkard Polster. “Toroidal circle planes that are not Minkowski planes”. *J. Geom.* 63 (1998), 154–167.

[PS01] Burkard Polster and Günter F. Steinke. *Geometries on surfaces.* Encyclopedia of Mathematics and its Applications, 84. Cambridge University Press, Cambridge, 2001, xxii+490.

[Sal67] Helmut Salzmann. “Topological planes”. *Advances in Math.* 2 (1967), 1–60.

[Sal+95] Helmut Salzmann et al. *Compact projective planes.* De Gruyter Expositions in Mathematics, 21. Walter de Gruyter & Co., Berlin, 1995, xiv+688.

[Sch80] A. Schenkel. *Topologische Minkowski-Ebenen.* Dissertation. Erlangen-Nürnberg, 1980.

[Ste84] Günter F. Steinke. “The automorphism group of locally compact connected topological Benz planes”. *Geom. Dedicata* 16 (1984), 351–357.

[Ste85] Günter F. Steinke. “Some Minkowski planes with 3-dimensional automorphism group”. *J. Geom.* 25. no. 1 (1985), 88–100.

[Ste04] Günter F. Steinke. “A family of flat Minkowski planes admitting 3-dimensional simple groups of automorphisms”. *Adv. Geom.* 4. no. 3 (2004), 319–340.

[Ste17] Günter F. Steinke. “Modified classical flat Minkowski planes”. *Adv. Geom.* 17. no. 3 (2017), 379–396.

[Str70] Karl Strambach. “Sphärische Kreisebenen”. *Math. Z.* 113 (1970), 266–292.