WEAK APPROXIMATION VERSUS THE HASSE PRINCIPLE FOR SUBVARIETIES OF ABELIAN VARIETIES

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Abstract. For varieties over global fields, weak approximation in the space of adelic points can fail. For a subvariety of an abelian variety one expects this failure is always explained by a finite descent obstruction, in the sense that the rational points should lie dense in the set of unobstructed (modified) adelic points. We show that this follows from a priori weaker assumptions concerning descent obstructions to the Hasse principle, i.e., to the existence of rational points. We also prove a similar statement for the obstruction coming from the Mordell-Weil sieve.

1. Introduction

For a smooth, projective and geometrically irreducible variety X over a number field k we consider the set $X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ of (modified) adelic points surviving descent by all X-torsors under finite group schemes over k. This is a closed subset of $X(\mathbb{A}_k)_\bullet$ containing the set of rational points $X(k)$. Here $X(\mathbb{A}_k)_\bullet$ is the set of connected components of the adelic points $X(\mathbb{A}_k) = \prod_v X(k_v)$, endowed with the quotient topology and we identify $X(k)$ with its injective image in $X(\mathbb{A}_k)_\bullet$.

Consider the following statements, in which $\overline{X(k)}$ is used to denote the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$.

$$\text{HP}^{\text{f-desc}}: \qquad \text{If } X(k)=\emptyset, \text{ then } X(\mathbb{A}_k)_\bullet^{\text{f-desc}}=\emptyset.$$

$$\text{WA}^{\text{f-desc}}: \qquad X(\mathbb{A}_k)_\bullet^{\text{f-desc}} = \overline{X(k)}\,.$$

One says that the finite descent obstruction is the only obstruction to the Hasse principle (resp. to weak approximation) when $\text{HP}^{\text{f-desc}}$ holds (resp. when $\text{WA}^{\text{f-desc}}$ holds).

For $X \subset A$ a subvariety of an abelian variety over a global field k we also consider analogous statements for the Mordell-Weil sieve, $X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} \subset A(\mathbb{A}_k)_\bullet$.

$$\text{HP}^{\text{MW}}: \qquad \text{If } X(k)=\emptyset, \text{ then } X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = \emptyset\,.$$

$$\text{WA}^{\text{MW}}: \qquad X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = \overline{X(k)}\,.$$

For a given variety, or class of varieties, it is clear that $\text{WA}^{\text{f-desc}}$ implies $\text{HP}^{\text{f-desc}}$ and similarly for WA^{MW} and HP^{MW}. The following theorems give converse results for subvarieties of abelian varieties. More precise results are proved in Corollary 2.2 and Theorem 4.1 below.

Theorem 1.1. *If* $\text{HP}^{\text{f-desc}}$ *holds for all subvarieties of abelian varieties over number fields, then* $\text{WA}^{\text{f-desc}}$ *holds for all subvarieties of abelian varieties over number fields.*

Theorem 1.2. *Let A/k be an abelian variety over a global field k. If* HP^{MW} *holds for all closed subvarieties of A, then* WA^{MW} *holds for all closed subvarieties of A.*

For a subvariety $X \subset A$ of an abelian variety, the sets appearing in these statements and the set of adelic points cut out by the Brauer-Manin obstruction are related by the containments below (where $X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ is only considered in the number field case)

$$\overline{X(k)} \subseteq X(\mathbb{A}_k)_\bullet^{\text{f-desc}} \subseteq X(\mathbb{A}_k)_\bullet^{\text{Br}} \subseteq X(\mathbb{A}_k)_\bullet \cap A(\mathbb{A}_k)_\bullet^{\text{Br}} \overset{\star}{\supseteq} X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} \supseteq \overline{X(k)} \tag{1.1}$$

It is conjectured that all of the sets appearing in (1.1) are equal (See [Sko01, p. 133] and [Sto07, Section 8] in the case of a curve inside its Jacobian, and [Poo06, 7.4] and in [PV10, Conjecture C] in general). The containment $\star$ is known to be an equality if the Tate-Shafarevich group of A has no nontrivial divisible elements [Sch99]. The equality of all other sets is known to hold if in addition $A(k)$ is finite [Sch99, Sto07]. In the function field case, equality of all sets in (1.1) is known for coset free subvarieties of abelian varieties which have no isotrivial isogeny factors and finite separable p-torsion [PV10] and for nonisotrivial curves of genus ≥ 2 [CV].

The weaker conjectures that HP^{MW} and $\text{HP}^{\text{f-desc}}$ hold for subvarieties of abelian varieties are supported by a wealth of empirical evidence as well as by a heuristic of Poonen [Poo06]. (The results there are stated for a curve in its Jacobian, but the argument only relies on the fact that X has positive codimension in A.) Theorem 1.2 shows that this same heuristic supports the stronger conjecture WA^{MW} as well.

A statement similar to Theorem 1.1 has been observed for curves over number fields by Stoll [Sto07, Section 8] in which case it is also closely related to a well known result concerning the section conjecture in anabelian geometry which has its origins in work of Nakamura and Tamagawa. Those arguments rely on Faltings' Theorem that $X(k)$ is finite for curves of genus at least 2 over number fields and so do not immediately generalize. Our proof relies instead on Theorem 2.1 below which provides a description of the topology on $X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ in terms of torsors over X. This also provides further insight into the connection between $\text{WA}^{\text{f-desc}}$ and the section conjecture, specifically Proposition 3.1.

We expect that with a suitable theory of finite (nonabelian) descent obstructions in the function field case, the analogue of Theorem 1.1 could be proved using methods similar to those here, but we will not attempt to do so.

2. The topology on $X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$

In this section we will prove the following theorem.

Theorem 2.1. *Suppose $X \subset T$ is a subvariety of a torsor under an abelian variety over a number field k. Let $\mathcal{B}$ be the collection of subsets of $X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ of the form $U_f = f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}})$, where $f : X' \to X$ is the pullback of a geometrically connected torsor $T' \to T$ under a finite group scheme over k. Then $\mathcal{B}$ is a basis for the subspace topology on $X(\mathbb{A}_k)_\bullet^{\text{f-desc}} \subset T(\mathbb{A}_k)_\bullet$.*

Proof. This follows immediately from Lemma 2.6 and Lemma 2.7 below. □

Corollary 2.2. *The following are equivalent for $X \subset T$ a subvariety of a torsor under an abelian variety over a number field k.*

(1) $\text{WA}^{\text{f-desc}}$ *holds for X;*
(2) $\text{HP}^{\text{f-desc}}$ *holds for every X'/k which arises as the pullback of a geometrically connected torsor $T' \to T$ under a finite group scheme over k;*

(3) $\mathrm{WA}^{\text{f-desc}}$ *holds for every X'/k which arises as the pullback of a geometrically connected torsor $T' \to T$ under a finite group scheme over k.*

One can easily deduce Theorem 1.1 from Corollary 2.2 using the following observation.

Remark 2.3. *Any torsor under an abelian variety can be embedded in an abelian variety by the following construction which was originally suggested to us by Poonen. If T is a torsor under an abelian variety A over k and L/k is a finite extension such that $T(L) \neq \emptyset$, the torsor structure determines an isomorphism $T_L \simeq A_L$ of varieties over L (up to translation). Restriction of scalars then gives a closed immersion $T \to \mathrm{Res}_{L/k}(T_L) \simeq \mathrm{Res}_{L/k}(A_L)$ identifying T as a closed subvariety of the abelian variety $\mathrm{Res}_{L/k}(A_L)$.*

Remark 2.4. *The proof of Theorem 2.1 shows that it and Corollary 2.2 also hold if one restricts to those $T' \to T$ which are twists of the multiplication n map on the abelian variety, in which case T and T' are torsors under the same abelian variety.*

Proof of Corollary 2.2. We first show that (2) implies (1). Let $x \in X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$. It suffices to show that every open subset $U \subset X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ which contains x also contains a k-point. By Theorem 2.1, there is an open subset of U of the from $U_f = f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}})$ containing x with $f : X' \to X$ the pullback of a geometrically connected torsor $T' \to T$. If we assume X' satisfies $\mathrm{HP}^{\text{f-desc}}$, then we have $X'(k) \neq \emptyset$ and so U contains a rational point.

To see that (1) implies (2) suppose that $f : X' \to X$ is as in (2) with $X'(\mathbb{A}_k)_\bullet^{\text{f-desc}} \neq \emptyset$. Then $U_f := f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}})$ is a nonempty open subset of $X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$. If X satisfies $\mathrm{WA}^{\text{f-desc}}$, then there must exist some k-rational point in U_f. The fiber above this point on X' is a finite subscheme $Z \subset X'$ with $Z(\mathbb{A}_k)_\bullet \cap X'(\mathbb{A}_k)_\bullet^{\text{f-desc}} \neq \emptyset$. By Remark 2.3 we can embed X' in an abelian variety B. Then, by functoriality of descent, we have that $Z(\mathbb{A}_k)_\bullet \cap B(\mathbb{A}_k)_\bullet^{\text{f-desc}} \neq \emptyset$. By [Sto07, Theorem 3.11] we then have that $Z(k) \neq \emptyset$. So $X'(k) \neq \emptyset$ showing that X' satisfies $\mathrm{HP}^{\text{f-desc}}$.

Clearly (3) implies (1), so to complete the proof it now suffices to show that (2) implies (3). Let $X' \to X$ be the pullback of $T' \to T$. Every torsor $X'' \to X'$ obtained by pulling back a geometrically connected torsor $T'' \to T'$ can be composed to give a torsor over X arising as pullback from a geometrically connected torsor over T. Thus, if (2) holds for $X \subset T$, then it must also hold for $X' \subset T'$ and we can conclude using the implication (2) implies (1) for $X' \subset T'$. ☐

Lemma 2.5. *Let A be an abelian variety over a global field k. For every open subset $U \subset A(\mathbb{A}_k)_\bullet$ containing the identity there exists an integer n such that $nA(\mathbb{A}_k)_\bullet \subset U$.*

Proof. Any open set in $A(\mathbb{A}_k)_\bullet$ is of the form $U = \prod_{v \in S} U_v \times \prod_{v \notin S} A(k_v)_\bullet$ with $U_v \subset A(k_v)_\bullet$ open and S finite. So it suffices to show that every open subset $U_v \subset A(k_v)_\bullet$ contains a set of the form $nA(k_v)$ for some n. For archimedean v this is clear since $A(k_v)_\bullet$ is a finite group. For nonarchimedean v, this follows from the fact that $A(k_v)$ contains a finite index torsion free pro-p-subgroup (where p is the residue characteristic of k_v). In the number field case this is Mattuck's theorem [Mat55] and in general follows from properties of the formal group of $A(k_v)$ (e.g., [Ser92, pp. 116-118]). ☐

Lemma 2.6. *Let $X \subset T$ be a subvariety of a torsor under an abelian variety over a number field k. The topology on $X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ generated by the subsets $U_f := f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}})$ with $f : X' \to X$ the pullback of a geometrically connected torsor $T' \to T$ under a finite abelian group scheme is at least as strong as the subspace topology $X(\mathbb{A}_k)_\bullet^{\text{f-desc}} \subset X(\mathbb{A}_k)_\bullet$.*

Proof. Suppose $x \in X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ and let $U \subset X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ be an open subset containing x. It is enough to find f such that $x \in U_f \subset U$. Then $U = V \cap X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ for some open subset $V \subset T(\mathbb{A}_k)_\bullet$.

Let A denote the abelian variety for which T is a torsor. The torsor structure on T gives a homeomorphism

$$T(\mathbb{A}_k)_\bullet \ni y \mapsto y - x \in A(\mathbb{A}_k)_\bullet\,,$$

sending x to the identity and V to a neighborhood $W \subset A(\mathbb{A}_k)_\bullet$ of the identity. By Lemma 2.5 there is an integer n such that $nA(\mathbb{A}_k)_\bullet \subset W$.

Let $g : T' \to T$ be an n-covering (i.e., a twist of the multiplication by n map on A) and let $f : X' \to X$ be the pullback to X. By [Sto07, Prop 5.17] we have

$$X(\mathbb{A}_k)_\bullet^{\text{f-desc}} = \bigcup_{\tau \in \mathrm{Sel}(f)} f^\tau(X'^\tau(\mathbb{A}_k)_\bullet^{\text{f-desc}})\,, \tag{2.1}$$

the union ranging over the subset $\mathrm{Sel}(f) \subset \mathrm{H}^1(k, G)$ parameterizing twists of X' that have adelic points. In light of (2.1) we can replace X' with a twist if needed and assume $x \in f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}})$.

Note that $f(X'(\mathbb{A}_k)_\bullet) \subset g(T'(\mathbb{A}_k)_\bullet) \subset V$ since the image of $g(T'(\mathbb{A}_k)_\bullet)$ in $A(\mathbb{A}_k)_\bullet$ is equal to $nA(\mathbb{A}_k)_\bullet$ which is contained in W by our assumption on n. So $x \in U_f := f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}}) \subset U = X(\mathbb{A}_k)_\bullet^{\text{f-desc}} \cap V$ as required. ☐

Lemma 2.7. *Let $X \subset T$ be a subvariety of a torsor under an abelian variety over a number field k and let $Y \to X$ be the pullback of a geometrically connected torsor $T' \to T$ under a finite abelian group scheme G/k. Then the set $U_f := f(Y(\mathbb{A}_k)_\bullet^{\text{f-desc}})$ is an open subset of $X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$.*

Proof. We must show that $U_f = U \cap X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ for some open subset $U \subset X(\mathbb{A}_k)$. By [Sto07, Prop. 5.17] we have

$$X(\mathbb{A}_k)_\bullet^{\text{f-desc}} = \bigcup_{\tau \in \mathrm{Sel}(f)} f^\tau(Y^\tau(\mathbb{A}_k)_\bullet^{\text{f-desc}})\,,$$

where $f^\tau : Y^\tau \to X$ denote the twists of f ranging over the subset $\mathrm{Sel}(f) \subset \mathrm{H}^1(k, G)$ parameterizing twists of Y that have adelic points.

Since $\mathrm{Sel}(f)$ is finite (See [HS02, Proposition 4.4] or [Poo17, 8.4.6]), there is a finite subset $S \subset \Omega_k$ of places of k such that for any $\tau, \tau' \in \mathrm{Sel}(f)$ we have that $\mathrm{res}_v(\tau)$ and $\mathrm{res}_v(\tau')$ differ at some prime $v \in S$ or they agree for all places $v \in \Omega_k$. For each $v \in S$, the subsets $f^\tau(Y^\tau(k_v)) \subset X(k_v)$ are closed, being the continuous image of a compact set in a Hausdorff space. Moreover, for varying τ these sets are either pairwise equal or disjoint, so

$$W_v := \left(\bigcup_{\tau \in \mathrm{Sel}(f)} f^\tau(Y^\tau(k_v)) \right) \setminus f(Y(k_v))$$

is a closed subset of $X(k_v)$ whose complement W_v^c contains $f(Y(k_v))$ and is disjoint from $f^\tau(Y^\tau(k_v))$ for any $\tau \in \mathrm{Sel}(f)$ such that $\mathrm{res}_v(\tau) \neq 0$. Let $U := \prod_{v \in S} W_v^c \times \prod_{v \notin S} X(k_v)$. This is an open subset of $X(\mathbb{A}_k)$ with the property that $U \cap X(\mathbb{A}_k)_\bullet^{\text{f-desc}} = f(Y(\mathbb{A}_k)_\bullet^{\text{f-desc}}) = U_f$. ☐

The following strengthening of Lemma 2.7 is not needed to prove Theorem 2.1, but will be used in the following section.

Lemma 2.8. *Let $f : X' \to X$ be a finite étale morphism of varieties over a number field k. Then $f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}}) \subset X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ is open.*

Proof. The proof of the preceding lemma shows that $f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}}) \subset X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ is open for any torsor $f : X' \to X$ under a finite group scheme over k. If $f : X' \to X$ is merely a finite étale morphism (but not necessarily a torsor), then we use that there is a torsor $g : X'' \to X$ under a finite group scheme which factors as the composition of f with a torsor $h : X'' \to X'$ [Sza09, Proposition 5.3.9]. By the previous lemma the sets $g^\tau(X''^\tau(\mathbb{A}_k)_\bullet^{\text{f-desc}}) \subset X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ are open for any twist of g. Thus by [Sto07, Prop. 5.17] applied to $h : X'' \to X'$, we see that $f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}}) = \bigcup_{\tau \in \mathrm{Sel}(h)} (f \circ h^\tau)(X''^\tau(\mathbb{A}_k)_\bullet^{\text{f-desc}})$ is a union of open sets and hence open. □

3. The Section Conjecture

Suppose X is a smooth, proper, geometrically irreducible variety over a number field k. By functoriality, any k-rational point determines a section of the fundamental exact sequence,

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to \pi_1(\mathrm{Spec}(k)) \to 1\,, \tag{3.1}$$

well defined up to conjugation by an element of $\pi_1(\overline{X})$. Grothendieck's section conjecture asserts that when X is a hyperbolic curve the map $X(k) \to \mathrm{Sec}(X/K)$ is a bijection. It is known that the full section conjecture for X follows if one assumes a weaker form of the section conjecture for all geometrically connected étale coverings $X' \to X$, namely that the existence of a section in $\mathrm{Sec}(X'/k)$ implies the existence of a k-rational point on X' [Sti13, Theorem 54 and Section 9.4]. Corollary 2.2 is an analogue of this result for adelic points surviving descent.

A section is called Selmer if for every place v of k its restriction to the decomposition group at v arises from a k_v-point. When $X \subset A$ is a subvariety of an abelian variety such a k_v-point (if it exists) is unique (up to the usual caveat at archimedean places) by [Sti13, Proposition 73]. This defines a map $\mathrm{loc} : \mathrm{Sec}^{\mathrm{Sel}}(X/k) \to X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ from the set of Selmer sections (considered up to $\pi_1(\overline{X})$-conjugacy) to adelic points surviving descent. This map is surjective by [HS12, Theorem 2.1] (See also [Sti13, Theorem 144] and [Sto07, Remark 8.9]). Surjectivity of loc shows that the section conjecture for X implies $\mathrm{WA}^{\text{f-desc}}$ for X. It is shown in [BKL, Proposition 2.13] that $\mathrm{WA}^{\text{f-desc}}$ is equivalent to the Selmer section conjecture, which asserts that $X(k) \to \mathrm{Sec}^{\mathrm{Sel}}(X/k)$ is a bijection.

The space $\mathrm{Sec}(X/k)$ of sections up to conjugacy is naturally equipped with a prodiscrete topology, inducing a subspace topology on $\mathrm{Sec}^{\mathrm{Sel}}(X/k)$. Theorem 2.1 implies the following.

Proposition 3.1. *Suppose $X \subset A$ is a closed subvariety of an abelian variety over a number field k. The surjective map $\mathrm{loc} : \mathrm{Sec}^{\mathrm{Sel}}(X/k) \to X(\mathbb{A}_k)_\bullet^{\text{f-desc}}$ is continuous and open.*

Proof. The sets

$$U_{X'} := \mathrm{image}\,(\mathrm{Sec}(X'/k) \to \mathrm{Sec}(X/k))$$

as $X' \to X$ ranges over geometrically connected étale coverings of X form a basis for the topology on $\mathrm{Sec}(X/k)$ [Sti13, Section 4.2]. The sets $V_{X'} := U_{X'} \cap \mathrm{Sec}^{\mathrm{Sel}}(X/k)$ thus form a basis for the subspace topology on $\mathrm{Sec}^{\mathrm{Sel}}(X/k) \subset \mathrm{Sec}(X/k)$. By [BKL, Lemma 2.15] (which is stated for an étale covering of hyperbolic curves, but whose proof works for any étale covering of geometrically connected varieties) we have that $V_{X'} = \mathrm{image}\left(\mathrm{Sec}^{\mathrm{Sel}}(X'/k) \to \mathrm{Sec}^{\mathrm{Sel}}(X/k)\right)$. Thus for any étale $f : X' \to X$ we have

(1) $\mathrm{loc}(V_{X'}) = f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}})$, and
(2) $\mathrm{loc}^{-1}(f(X'(\mathbb{A}_k)_\bullet^{\text{f-desc}})) = V_{X'}$.

Lemma 2.8 together with (1) shows that loc is open. Lemma 2.6 together with (2) shows that Sec is continuous. □

4. The Mordell-Weil Sieve

Theorem 4.1. *Let $X \subset A$ be a closed subvariety of an abelian variety over a global field k. If $\mathrm{HP}^{\mathrm{MW}}$ holds for every $Y \subset A$ which is the pullback to X of a morphism $\rho : A \to A$ of the form $\rho(a) = na + P$ with $n \geq 1$ and $P \in A(k)$, then $\mathrm{WA}^{\mathrm{MW}}$ holds for X.*

Proof. Let $X \subset A$ be a closed subvariety and suppose $x \in X(\mathbb{A}_k)_\bullet \cap \overline{A(k)}$. Let $U \subset X(\mathbb{A}_k)_\bullet$ be an open subset containing x. It suffices to show that $U \cap X(k)$ is nonempty. Let $V \subset A(\mathbb{A}_k)_\bullet$ be an open subset such that $U = V \cap X(\mathbb{A}_k)_\bullet$. By Lemma 2.5 there is an integer n_0 such that $(x + n_0 A(\mathbb{A}_k)_\bullet) \subset V$

Since $x \in \overline{A(k)}$ there exists a sequence of points $P_n \in A(k)$ such that

(1) $x \in P_n + nA(\mathbb{A}_k)_\bullet$ for all $n \geq 1$, and
(2) $P_{mn} \equiv P_n \bmod nA(k)$, for all $m, n \geq 1$.

The existence of P_n satisfying (1) follows from Lemma 2.5. That these P_n can be chosen to also satisfy (2) follows from König's Lemma. This same argument shows as in [PV10, Theorem E] that the inclusion $A(k) \to A(\mathbb{A}_k)_\bullet$ induces an isomorphism $\widehat{A(k)} \simeq \overline{A(k)}$ between the profinite completion of $A(k)$ and its topological closure in $A(\mathbb{A}_k)_\bullet$.

For $m \geq 1$, let $R_m \in A(k)$ be such that $P_{mn_0} - P_{n_0} = n_0 R_m$. The possible choices for R_m differ by an element in $A(k)[n]$. Since $A(k)_{\mathrm{tors}}$ is finite we can choose the R_m so that for any $\ell, m \geq 1$ we have $R_{\ell m} \equiv R_m \bmod mA(k)$. With such a choice, the limit $y := \lim_{m\to\infty} R_{m!} \in \overline{A(k)}$ exists. Let $\rho : A \to A$ be the morphism given by $\rho(a) = n_0 a + P_{n_0}$. Since $\rho : A(\mathbb{A}_k)_\bullet \to A(\mathbb{A}_k)_\bullet$ is continuous, we have $\rho(y) = \rho(\lim_m R_{m!}) = \lim_m n_0 R_{m!} + P_{n_0} = \lim_m P_{n_0 m!} = x$.

Let $Y \to X$ be the pullback of ρ to X. Then $Y \subset A$ and from above we have that $y \in Y(\mathbb{A}_k)_\bullet \cap \overline{A(k)}$. Assuming $\mathrm{HP}^{\mathrm{MW}}$ holds for Y we find $Y(k) \neq \emptyset$. Since $\rho(Y(\mathbb{A}_k)_\bullet) \subset U$, this shows that $U \cap X(k) \neq \emptyset$. □

Under the additional assumption that $X(k)$ is finite we have the converse to Theorem 4.1.

Proposition 4.2. *Suppose $X \subset A$ is a subvariety of an abelian variety over a global field with $X(k)$ finite. If $\mathrm{WA}^{\mathrm{MW}}$ holds for X, then $\mathrm{WA}^{\mathrm{MW}}$ holds for every $Y \subset A$ obtained as the pullback of a morphism $\rho : A \to A$ of the form $\rho(a) = na + b$ with $n \geq 1$ and $b \in A(k)$.*

Proof. Let $y \in Y(\mathbb{A}_k)_\bullet \cap \overline{A(k)}$. Then $x = \rho(y) \in X(\mathbb{A}_k)_\bullet \cap \overline{A(k)} = \overline{X(k)} = X(k)$ since X satisfies $\mathrm{WA}^{\mathrm{MW}}$ and $X(k)$ is finite. The zero-dimensional subscheme $Z := \rho^{-1}(x) \subset Y \subset A$ satisfies $\mathrm{WA}^{\mathrm{MW}}$ by [Sto07, Theorem 3.11] in the number field case and by [PV10, Theorem E] and [CV, Proposition 3.1] in the function field case. Since $y \in Z(\mathbb{A}_k)_\bullet \cap \overline{A(k)}$ we conclude that $y \in Y(k)$. □

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