ON THE LOCAL-GLOBAL PRINCIPLE FOR DIVISIBILITY IN THE COHOMOLOGY OF ELLIPTIC CURVES

BRENDAN CREUTZ

ABSTRACT. For every prime power p^n with $p = 2$ or 3 and $n \geq 2$ we give an example of an elliptic curve over $\mathbb{Q}$ containing a rational point which is locally divisible by p^n but is not divisible by p^n. For these same prime powers we construct examples showing that the analogous local-global principle for divisibility in the Weil-Châtelet group can also fail.

1. INTRODUCTION

Let G be a connected commutative algebraic group over a number field k, and let n and r be nonnegative integers. An element ρ in the Galois cohomology group $\mathrm{H}^r(k, G) := \mathrm{H}^r(\mathrm{Gal}(\overline{k}/k), G(\overline{k}))$ is divisible by n if there exists $\rho' \in \mathrm{H}^r(k, G)$ such that $n\rho' = \rho$. We say ρ is locally divisible by n if, for all primes v of k, there exists $\rho'_v \in \mathrm{H}^r(k_v, G)$ such that $n\rho'_v = \mathrm{res}_v(\rho)$. It is natural to ask whether every element locally divisible by n is necessarily divisible by n. When the answer is yes, we say the local-global principle for divisibility by n holds.

For $r = 0$ and $G = \mathbb{G}_m$, the answer is given by the Grunwald-Wang theorem (see [NSW08, IX.1]); the local-global principle for divisibility by n holds, except possibly when 8 divides n. The case $r = 1$ and $G = \mathbb{G}_m$ is trivial in light of Hilbert's theorem 90. For $r \geq 2$ and general G, a result of Tate implies that the local-global principle for divisibility by n always holds (see Theorem 2.1 below).

A study of the problem for $r = 0$ and general G was initiated by Dvornicich and Zannier in [DZ01], with particular focus on elliptic curves in [DZ04, DZ07, PRV12]. For elliptic curves over $\mathbb{Q}$, their results show that the local-global principle for divisibility by a prime power p^n holds for $n = 1$ or $p \geq 11$, and they have constructed counterexamples for $p^n = 4$.[1] For $r = 1$ and G an elliptic curve, the question was in effect raised by Cassels [Cas62a, Problem 1.3]. In particular, he asked whether elements of $\mathrm{H}^1(k, G)$ that are everywhere locally trivial must be divisible. In response, Tate proved the local-global principle for divisibility by a prime p [Cas62b]. Cassels' question is considered again in [Baš72], and recently by Çiperiani and Stix [ÇS13] who showed that, for elliptic curves over $\mathbb{Q}$, the local-global principle for divisibility by p^n holds for all prime powers with $p \geq 11$. An example showing that it does not hold in general over $\mathbb{Q}$ for any $p^n = 2^n$ with $n \geq 2$ was constructed in [Cre13].

In this note we produce examples settling these questions for the remaining undecided powers of the primes 2 and 3. We prove the following.

Theorem. *Let $n \geq 2$ be an integer, let $p \in \{2, 3\}$ and let $r \in \{0, 1\}$. Then there exists an elliptic curve E over $\mathbb{Q}$ for which the local-global principle for divisibility by p^n fails in $\mathrm{H}^r(\mathbb{Q}, E)$.*

[1] Paladino et al. have recently announced a proof of the local-global principle for powers of 5 and 7 [PRV].

Acknowledgements. It is a pleasure to thank Jakob Stix for a number of helpful comments, including pointing out case (3) in Theorem 2.1.

Notation. Throughout the paper p denotes a prime number, m and n are a positive integers, and r is a nonnegative integer. As above, G is a connected commutative algebraic group defined over a number field k with a fixed algebraic closure $\overline{k}$. We will use K to denote a field containing k and use $\overline{K}$ to denote a fixed algebraic closure of K containing $\overline{k}$. For a $\mathrm{Gal}(\overline{k}/k)$-module M, let $M^\vee$ denote its Cartier dual and define

$$Ш^r(k,M) := \ker\left(\mathrm{H}^r(k,M) \xrightarrow{\prod \mathrm{res}_v} \prod_v \mathrm{H}^r(k_v,M)\right),$$

the product running over all primes of k.

2. The obstruction to the local-global principle for divisibility

Because K has characteristic 0, multiplication by n is a finite étale endomorphism of G. Hence, for any $r \geq 0$, the short exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules

$$0 \to G[n] \xrightarrow{\iota} G \xrightarrow{n} G \to 0$$

gives rise to an exact sequence,

$$\mathrm{H}^r(K,G[n]) \xrightarrow{\iota_*} \mathrm{H}^r(K,G) \xrightarrow{n_*} \mathrm{H}^r(K,G) \xrightarrow{\delta_n} \mathrm{H}^{r+1}(K,G[n]) \xrightarrow{\iota_*} \mathrm{H}^{r+1}(k,G)\,. \tag{2.1}$$

From this one easily sees that an element $\rho \in \mathrm{H}^r(k,G)$ is locally divisible by n if and only if $\delta_n(\rho) \in Ш^{r+1}(k,G[n])$, and that ρ is divisible by n if and only if $\delta_n(\rho) = 0$. In particular, the local-global principle for divisibility by n in $\mathrm{H}^r(k,G)$ holds whenever $Ш^{r+1}(k,G[n]) = 0$. Combining this observation with Tate's duality theorems yields the following.

Theorem 2.1. *Assume any of the following:*

(1) $r = 0$ *and* $Ш^1(k,G[n]) = 0$*;*
(2) $r = 1$ *and* $Ш^1(k,G[n]^\vee) = 0$*; or*
(3) $r \geq 2$*.*

Then the local-global principle for divisibility by n *in* $\mathrm{H}^r(k,G)$ *holds.*

Proof. As noted above, in each case it suffices to show that $Ш^{r+1}(k,G[n]) = 0$. Case (1) is trivial, and cases (2) and (3) follow immediately from [Tat63, Theorem 3.1]. □

The following proposition shows that when G is a principally polarized abelian variety, the conditions in the theorem are necessary, at least conjecturally.

Proposition 2.2. *Suppose* G *is an abelian variety with dual* $G^\vee$*. Then for every* $\xi \in Ш^1(k,G[n])$*, exactly one of the following hold:*

(1) $\xi = 0$*;*
(2) $\xi = \delta_n(\rho)$ *for some* $\rho \in G(k)$ *that is locally divisible by* n*, but is not divisible by* n*; or*
(3) $\iota_*(\xi) \neq 0$*, in which case there either exists* $\rho \in Ш^1(k,G^\vee)$ *such that* ρ *is not divisible by* n*, or* $\iota_*(\xi)$ *is divisible in* $Ш^1(k,G)$ *by all powers of* n*.*

If G *is a principally polarized abelian variety and* $Ш^1(k,G)$ *is finite, then the local-global principle for divisibility by* n *holds in* $\mathrm{H}^r(k,G)$ *for every* $r \geq 0$ *if and only if* $Ш^1(k,G[n]) = 0$*.*

Proof. Exactness of (2.1) implies that the cases in the first statement of the proposition are exhaustive and mutually exclusive. For the claim in case (3) we may apply [Cre13, Thm 3], which states that $Ш^1(k, G^\vee) \subset n\,\mathrm{H}^1(k, G^\vee)$ if and only if the image of $\iota_* : Ш^1(k, G[n]) \to Ш^1(k, G)$ is contained in the maximal divisible subgroup of $Ш^1(k, G)$.

Now suppose G is a principally polarized and that $Ш^1(k, G)$ is finite. We must prove the equivalence in the second statement. One direction follows from Theorem 2.1 since $G[n] = G[n]^\vee$. The other direction follows from the first statement in the proposition, since finiteness of $Ш^1(k, G)$ implies that it contains no nontrivial divisible elements as in case (3). □

The next lemma formalizes a method for constructing elements of $Ш^1(k, G[mn])$, for some $m \geq 1$.

Lemma 2.3. *Let $m \geq 1$ and let $j : G[n] \subset G[mn]$ be the inclusion map. Suppose $\xi \in \mathrm{H}^1(k, G[n])$ is such that $\mathrm{res}_v(\xi) \in \delta_n(G(k_v)[m])$, for all primes v of k. Then*

(1) $j_*(\xi) \in Ш^1(k, G[mn])$;
(2) $j_*(\xi) = 0$ if and only if $\xi \in \delta_n(G(k)[m])$;
(3) if $\xi = \delta_n(\rho)$ for some $\rho \in G(k)$, then $m\rho$ is locally divisible by mn; and
(4) if $\xi = \delta_n(\rho)$ for some $\rho \in G(k)$ and $j_*(\xi) \neq 0$, then $m\rho$ is not divisible by mn.

Proof. The connecting homomorphism $G(K)[m] \to \mathrm{H}^1(k, G[n])$ arising from the short exact sequence

$$0 \to G[n] \xrightarrow{j} G[mn] \xrightarrow{n} G[m] \to 0$$

is the restriction of the δ_n to $G(K)[m]$. This implies that

$$\ker\left(j_* : \mathrm{H}^1(K, G[n]) \to \mathrm{H}^1(K, G[mn])\right) = \delta_n(G(K)[m])\,,$$

from which the first two statements in the proposition easily follow.

The inclusion $j : G[n] \subset G[mn]$ also induces a commutative diagram

$$\begin{array}{ccccccc} G(K)[n] & \hookrightarrow & G(K) & \xrightarrow{n} & G(K) & \xrightarrow{\delta_n} & \mathrm{H}^1(K, G[n]) \\ \downarrow{\scriptstyle j} & & \| & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle j_*} \\ G(K)[mn] & \hookrightarrow & G(K) & \xrightarrow{mn} & G(K) & \xrightarrow{\delta_{mn}} & \mathrm{H}^1(K, G[mn])\,, \end{array} \tag{2.2}$$

where the rows are the exact sequence (2.1) with $r = 0$, and the same sequence with mn in place of n. From this the last two statements can be deduced easily. □

3. The examples for $p = 2$

Proposition 3.1. *Let E be the elliptic curve defined by $y^2 = (x + 2795)(x - 1365)(x - 1430)$ and let $P = (341 : 59136 : 1) \in E(\mathbb{Q})$. For every $n \geq 2$, the point $2^{n-1}P$ is locally divisible by 2^n, but not divisible by 2^n. In particular, the local-global principle for divisibility by 2^n in $E(\mathbb{Q})$ fails for every $n \geq 2$.*

Remark 3.2. *This example was constructed by Dvornicich and Zannier who proved the proposition in the case $n = 2$* [DZ04, §4]. *Using Lemma 2.3 their arguments apply to all $n \geq 2$. We include our own proof here since our examples for $p = 3$ will be obtained using a similar, though more involved argument.*

Proof. Fix the basis $P_1 = (1365 : 0 : 1)$, $P_2 = (1430 : 0 : 1)$ for $E[2]$. By [Sil86, Proposition X.1.4] the composition of δ_2 with isomorphism $\mathrm{H}^1(K, E[2]) \simeq (K^\times/K^{\times 2})^2$ is given explicitly by

$$Q = (x_0, y_0) \longmapsto \begin{cases} (x_0 - 1365, x_0 - 1430) & \text{if } Q \neq P_1, P_2 \\ (-1, -65) & \text{if } Q = P_1 \\ (65, 65) & \text{if } Q = P_2 \\ (1, 1) & \text{if } Q = 0 \end{cases}.$$

In particular, $\delta_2(P) = (-1, -1)$ and $\delta_2(E(K)[2])$ is generated by $\{(-1, -65), (65, 65)\}$. It follows that $\delta_2(P) \in \delta_2(E(K)[2])$ if and only if at least one of 65, -65 or -1 is a square in K. If $K = \mathbb{Q}_v$ for some $v \leq \infty$, then one of these is a square. Indeed, 65 is a square in $\mathbb{R}$ and in $\mathbb{Q}_2$, -1 is a square $\mathbb{Q}_5$ and in $\mathbb{Q}_{13}$, and for all other primes v the Legendre symbols satisfy the identity $\left(\frac{-1}{v}\right)\left(\frac{65}{v}\right) = \left(\frac{-65}{v}\right)$. Hence $\xi := \delta_2(P)$ satisfies the hypothesis of Lemma 2.3 with (m, n) replaced by $(2^{n-1}, 2)$.

On the other hand, 65, -65 and -1 are not squares in $\mathbb{Q}$, and $E(\mathbb{Q})[2^\infty] = E(\mathbb{Q})[2]$ (the reduction mod 3 is nonsingular, so the 2-primary torsion must inject into the group of $\mathbb{F}_3$-points on the reduced curve. This group has order less than 8 by Hasse's theorem). So the result follows from Lemma 2.3. □

Proposition 3.3. *Let E be the elliptic curve defined by $y^2 = x(x + 80)(x + 205)$. Then $Ш^1(\mathbb{Q}, E) \not\subset 4\,\mathrm{H}^1(\mathbb{Q}, E)$. In particular, the local-global principle for divisibility by 2^n in $\mathrm{H}^1(\mathbb{Q}, E)$ fails for every $n \geq 2$.*

Proof. This is [Cre13, Theorem 5]; we are content to sketch the proof. Much like the previous proof, one uses the explicit description of the map $\delta_2 : E(K) \to \mathrm{H}^2(K, E[2]) \simeq (K^\times/K^{\times 2})^2$ to show that there is an element $\xi \in \mathrm{H}^1(\mathbb{Q}, E[2]) \setminus \delta_2(E(\mathbb{Q}))$ which maps into $\delta_2(E(\mathbb{Q}_v))$ everywhere locally. Lemma 2.3 then shows that the image of ξ in $\mathrm{H}^1(k, E[4])$ falls under case (3) of Proposition 2.2. This gives the result, since $Ш^1(\mathbb{Q}, E)[2^\infty]$ is finite (as one can check in multiple ways, with or without the assistance of a computer). □

4. Diagonal cubic curves and 3-coverings

The examples for $p = 2$ were constructed using an explicit description of the map

$$E(K) \xrightarrow{\delta_2} \mathrm{H}^1(K, E[2]) \simeq \left(K^\times/K^{\times 2}\right)^2.$$

Another way to describe the connecting homomorphism is in the language of n-coverings. An n-covering of an elliptic curve E over K is a K-form of the multiplication by n map on E. In other words, an n-covering of E is a morphism $\pi : C \to E$ such that there exists an isomorphism $\psi : E_{\overline{K}} \to C_{\overline{K}}$ of the curves base changed to the algebraic closure $\overline{K}$ which satisfies $\pi \circ \psi = n$. We now summarize how this notion can be used to give an interpretation of the group $\mathrm{H}^1(K, E[n])$. Details may be found in [CFO+08, §1].

An isomorphism of n-coverings of E is, by definition, an isomorphism in the category of E-schemes. The automorphism group of the n-covering $n : E \to E$ can be identified with $E[n]$ acting by translations. By a standard result in Galois cohomology (the twisting principle) the K-forms of $n : E \to E$ are parameterized, up to isomorphism by $\mathrm{H}^1(K, E[n])$. Under this identification the connecting homomorphism δ_n sends a point $P \in E(K)$ to the

isomorphism class of the n-covering,

$$\pi_P : E \to E\,, \quad Q \mapsto nQ + P\,.$$

In particular, the isomorphism class of an n-covering $\pi : C \to E$ is equal to $\delta_n(P)$ if and only if $P \in \pi(C(K))$.

Our examples for $p = 3$ will come from elliptic curves of the form $E : x^3 + y^3 + dz^3 = 0$ with distinguished point $(1 : -1 : 0)$, where $d \in \mathbb{Q}^\times$. For these curves we can write down some of the 3-coverings quite explicitly. According to Selmer, the following lemma goes back to Euler (see [Sel51, Theorem 1]).

Lemma 4.1. *Let $E : x^3+y^3+dz^3 = 0$ and suppose $a, b, c \in \mathbb{Q}^\times$ are such that $abc = d$. Then the curve $C : aX^3 + bY^3 + cZ^3 = 0$ together with the map $\pi : C \to E$ defined by*

$$\begin{aligned} x + y &= 9abcX^3Y^3Z^3 \\ x - y &= (aX^3 - bY^3)(bY^3 - cZ^3)(cZ^3 - aX^3) \\ z &= 3(abX^3Y^3 + bcY^3Z^3 + caZ^3X^3)XYZ \end{aligned}$$

is a 3-covering of E.

Proof. A direct computation verifies that these equations define a nonconstant morphism $\pi : C \to E$, which, by virtue of the fact that E and C are smooth genus 1 curves, implies that it is finite and étale. The map $\psi : E_{\overline{K}} \to C_{\overline{K}}$ defined by

$$x = \sqrt[3]{a}X\,, \quad y = \sqrt[3]{b}Y\,, \quad z = \sqrt[3]{c/d}Z \tag{4.1}$$

is clearly an isomorphism. It is quite evident that $E[3]$, which is cut out by $xyz = 0$, is mapped by $\pi \circ \psi$ to the identity $(1 : -1 : 0) \in E_{\overline{K}}$. Therefore $\pi \circ \psi$ is an isogeny which factors through multiplication by 3. Since it has degree 9 it must in fact be multiplication by 3, and so π is a 3-covering. □

Lemma 4.2. *Suppose $d = 3d'$ and let $\xi \in \mathrm{H}^1(K, E[3])$ be the class corresponding to the 3-covering as in Lemma 4.1 with $C : X^3 + 3Y^3 + d'Z^3 = 0$. Then $\xi \in \delta_3(E(K)[3])$ if any of the following hold:*

(1) $3 \in K^{\times 3}$;
(2) $d' \in K^{\times 3}$;
(3) $3d \in K^{\times 3}$;
(4) $d \in K^{\times 3}$ and K contains the 9th roots of unity; or
(5) $d \in K^{\times 3}$ and K contains a cube root of unity ζ_3 such that $3\zeta_3 \in K^{\times 3}$.

Corollary 4.3. *Suppose $d = 3d'$ and let $\xi \in \mathrm{H}^1(\mathbb{Q}, E[3])$ be the class of the 3-covering in Lemma 4.2. Then $\mathrm{res}_v(\xi) \in \delta_3(E(\mathbb{Q}_v)[3])$, for every prime $v \nmid d$.*

Proof. Suppose $v \nmid d$ and set $K = \mathbb{Q}_v$. By assumption $d, d', 3$, and $3d$ are units and, since $\mathbb{Z}_v^\times/\mathbb{Z}_v^{\times 3}$ is cyclic, one of them must be a cube. Moreover, if $\mathbb{Q}_v$ does not contain a primitive cube root of unity, then they are all cubes (since $\mathbb{Z}_v^\times/\mathbb{Z}_v^{\times 3}$ is trivial in this case). In light of this, and the first three cases in the lemma, we may assume $d \in \mathbb{Q}_v^{\times 3}$ and that $\mathbb{Z}_v$ contains a primitive cube root of unity ζ_3. If ζ_3 is a cube, then case (4) of the lemma applies. If ζ_3 is not a cube, then the class of 3 is contained in the subgroup of $\mathbb{Q}_v^\times/\mathbb{Q}_v^{\times 3}$ generated by ζ_3, in which case (5) of the lemma applies. This establishes the corollary. □

Proof of Lemma 4.2. By the discussion at the beginning of this section, it suffices to show that in each of these cases there is a K-rational point on C which maps to a 3-torsion point on E.[2] The 3-torsion points are the intersections of E with the hyperplanes defined by $x = 0$, $y = 0$ and $z = 0$. In the first three cases (resp.) the points

$$(-\sqrt[3]{3} : 1 : 0)\,, \quad (-\sqrt[3]{d'} : 0 : 1)\,, \text{ and } \quad (0 : -\sqrt[3]{3d} : 3)$$

are defined over K, and the explicit formula for π given in Lemma 4.1 shows that they map to $(1 : -1 : 0) \in E(K)[3]$.

In case (4) K Contains a primitive 9th root of unity ζ_9 and a cube root $\sqrt[3]{d}$ of d. Then

$$\left((2\zeta_9^5 + \zeta_9^4 + \zeta_9^2 + 2\zeta_9)\sqrt[3]{d} : (-\zeta_9^3 + \zeta_9^2 + \zeta_9 - 1)\sqrt[3]{d} : -3\right) \in C(K)\,,$$

and one can check that it maps under π to the point $(0 : -\sqrt[3]{d} : 1)$. In case (5) K contains cube roots $\sqrt[3]{d}$ and $\beta = \sqrt[3]{3\zeta_3}$, where ζ_3 is a cube root of unity. One may check that $(\beta^2\sqrt[3]{d} : \beta\sqrt[3]{d} : -3) \in C(K)$, and that this point maps under π to the point $(\zeta^2 : -1 : 0)$. □

5. The examples for $p = 3$

Proposition 5.1. *Let $E : x^3 + y^3 + 30z^3 = 0$ be the elliptic curve over $\mathbb{Q}$ with distinguished point $P_0 = (1 : -1 : 0)$, and let $P = (1523698559 : -2736572309 : 826803945) \in E(\mathbb{Q})$. For every $n \geq 2$, $3^{n-1}P$ is locally divisible by 3^n, but not divisible by 3^n. In particular, the local-global principle for divisibility by 3^n in $E(\mathbb{Q})$ fails for every $n \geq 2$.*

Proof. Let $C : X^3 + 3Y^3 + 10Z^3$ be the 3-covering of E as in Lemma 4.1, and let $\xi \in \mathrm{H}^1(\mathbb{Q}, E[3])$ be the corresponding cohomology class. One may check that the point $Q = (-11 : 3 : 5) \in C(\mathbb{Q})$ maps to P. Thus $\xi = \delta_3(P)$. By Corollary 4.3, $\mathrm{res}_v(\xi) \in \delta_3(E(\mathbb{Q}_v)[3])$ for all primes $v \nmid 30$. Also, since $10 \in \mathbb{Q}_3^{\times 3}$ and 3 is a cube in both $\mathbb{Q}_2$ and $\mathbb{Q}_5$ the first two cases of Lemma 4.2 show that $\mathrm{res}_v(\xi) \in \delta_3(E(\mathbb{Q}_v)[3])$ also for $v \mid 30$. On the other hand, $\xi \neq 0$ because $C(\mathbb{Q})$ does not contain a point lying on the subscheme defined by $XYZ = 0$. Since, $E(\mathbb{Q})[3] = 0$ the result follows by applying Lemma 2.3. □

Remark 5.2. *For any $d \in \{51, 132, 159, 213, 219, 246, 267, 321, 348, 402, 435\}$ the same argument applies, giving more examples where the local-global principle for divisibilitiy by 3^n in $E(\mathbb{Q})$ fails for all $n \geq 2$.*

Proposition 5.3. *Let $d \in \{138, 165, 300, 354\}$ and let $E : x^3 + y^3 + dz^3 = 0$ be the elliptic curve over $\mathbb{Q}$ with distinguished point $P_0 = (1 : -1 : 0)$. Then $Ш^1(\mathbb{Q}, E) \not\subset 9\,\mathrm{H}^1(\mathbb{Q}, E)$. In particular, the local-global principle for divisibility by 3^n in $\mathrm{H}^1(\mathbb{Q}, E)$ fails for every $n \geq 2$.*

Proof. Set $d' = d/3$. Let $C : X^3 + 3Y^3 + d'Z^3$ be the 3-covering of E as in Lemma 4.1, and let $\xi \in \mathrm{H}^1(\mathbb{Q}, E[3])$ be the corresponding cohomology class. In all cases one easily checks that $d' \in \mathbb{Q}_3^{\times 3}$ and that $3 \in \mathbb{Q}_v^{\times 3}$ for all $v \mid d'$. So using the first two cases of Lemma 4.2 and Corollary 4.3 we see that $\mathrm{res}_v(\xi) \in \delta_3(E(\mathbb{Q}_v)[3])$ for every prime v. Then, by Lemma 2.3, the image of ξ in $\mathrm{H}^1(\mathbb{Q}, E[9])$ lies in $Ш^1(\mathbb{Q}, E[9])$.

For these values of d, Selmer showed that $E(\mathbb{Q}) = \{(1 : -1 : 0)\}$ and $C(\mathbb{Q}) = \emptyset$ [Sel51, Theorem IX and Table 4b]. The latter implies that the image of ξ in $Ш^1(\mathbb{Q}, E[3^n])$ is

[2]The points given below were found with the assistance of the `Magma` computer algebra system described in [BCP97]. A `Magma` script verifying the claims here can be found in the source file of the `arXiv` distribution of this article.

nontrivial for every $n \geq 2$. Moreover, Selmer's proof shows that $3Ш^1(\mathbb{Q}, E)[3^\infty] = 0$. In particular $Ш^1(\mathbb{Q}, E)[3^\infty]$ contains no nontrivial infinitely divisible elements. Thus we are in case (3) of Proposition 2.2, and conclude that there exists some element of $Ш^1(\mathbb{Q}, E)$ which is not divisible by 9 in $\mathrm{H}^1(\mathbb{Q}, E)$. □

Remark 5.4. *The argument in the proof above shows that $C \in Ш^1(\mathbb{Q}, E)$, but does not show that $C \notin 9\,\mathrm{H}^1(\mathbb{Q}, E)$. Rather, the elements of $Ш^1(\mathbb{Q}, E)$ which are proven not to be divisible by 9 in $\mathrm{H}^1(\mathbb{Q}, E)$ are those that are not orthogonal to C with respect to the Cassels-Tate pairing. See* [Cre13, Theorem 4].

References

[Baš72] M. I. Bašmakov, *Cohomology of Abelian varieties over a number field*, Uspehi Mat. Nauk **27** (1972), no. 6(168), 25–66 (Russian). MR0399110 (53 #2961)

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, DOI 10.1006/jsco.1996.0125. Computational algebra and number theory (London, 1993). MR1484478

[Cas62a] J. W. S. Cassels, *Arithmetic on curves of genus* 1. *III. The Tate-Šafarevič and Selmer groups*, Proc. London Math. Soc. (3) **12** (1962), 259–296. MR0163913 (29 #1212)

[Cas62b] ———, *Arithmetic on curves of genus* 1. *IV. Proof of the Hauptvermutung*, J. Reine Angew. Math. **211** (1962), 95–112. MR0163915 (29 #1214)

[ÇS13] Mirela Çiperiani and Jakob Stix, *Weil–Châtelet divisible elements in Tate–Shafarevich groups II: On a question of Cassels*, J. Reine Angew. Math. (2013). (to appear).

[CFO+08] J. E. Cremona, T. A. Fisher, C. O'Neil, D. Simon, and M. Stoll, *Explicit n-descent on elliptic curves. I. Algebra*, J. Reine Angew. Math. **615** (2008), 121–155, DOI 10.1515/CRELLE.2008.012. MR2384334 (2009g:11067)

[Cre13] Brendan Creutz, *Locally trivial torsors that are not Weil-Châtelet divisible*, Bull. Lond. Math. Soc., posted on 2013, DOI 10.1112/blms/bdt019, (to appear in print).

[DZ01] Roberto Dvornicich and Umberto Zannier, *Local-global divisibility of rational points in some commutative algebraic groups*, Bull. Soc. Math. France **129** (2001), no. 3, 317–338 (English, with English and French summaries). MR1881198 (2002k:14031)

[DZ04] ———, *An analogue for elliptic curves of the Grunwald-Wang example*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 1, 47–50, DOI 10.1016/j.crma.2003.10.034 (English, with English and French summaries). MR2038083 (2004k:11088)

[DZ07] ———, *On a local-global principle for the divisibility of a rational point by a positive integer*, Bull. Lond. Math. Soc. **39** (2007), no. 1, 27–34, DOI 10.1112/blms/bdl002. MR2303515 (2007k:14030)

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR2392026 (2008m:11223)

[PRV12] Laura Paladino, Gabriele Ranieri, and Evelina Viada, *On local-global divisibility by p^n in elliptic curves*, Bull. Lond. Math. Soc. **44** (2012), no. 4, 789–802, DOI 10.1112/blms/bds012. MR2967246

[PRV] ———, *On the minimal set for counterexamples to the local-global principle*, available at `arXiv:1107.3431v1`.

[Sel51] Ernst S. Selmer, *The Diophantine equation $ax^3 + by^3 + cz^3 = 0$*, Acta Math. **85** (1951), 203–362 (1 plate). MR0041871 (13,13i)

[Sil86] Joseph H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1986. MR817210 (87g:11070)

[Tat63] John Tate, *Duality theorems in Galois cohomology over number fields*, Proc. Internat. Congr. Mathematicians (Stockholm, 1962), Inst. Mittag-Leffler, Djursholm, 1963, pp. 288–295. MR0175892 (31 #168)

School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia
E-mail address: brendan.creutz@maths.usyd.edu.au
URL: http://magma.maths.usyd.edu.au/~bcreutz