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ARCS IN PROJECTIVE PLANES OVER PRIME FIELDS

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Let $p \neq 2$ be a prime number. We will work only over $PG(2,p)$, the projective plane over the field of p elements.

Recall that an arc $A \subseteq PG(2,p)$ is a set for which no three of its elements are collinear. An arc is called complete when it is not a proper subset of another arc. The simplest example of a complete arc is a conic which has $p+1$ elements. On the other hand, Segre has shown that if A is a complete arc which is not a conic then ([**3**], [**1**] Corollary, pg. 238)

$$\#A \leq p - \sqrt{p}/4 + 7/4 \qquad (*)$$

(This result is valid even when p is not a prime number). The purpose of this note is to improve (*) when $p \neq 2$ is a prime number. The proof follows closely Segre's proof of (*), the only novelty being the use of a sharper estimate for the number of rational points of a curve over a finite field which follows from the results of [**4**].

THEOREM. *Let $p \neq 2$ be a prime number and $A \subseteq PG(2,p)$ a complete arc, not a conic. Then*

$$\#A \leq p - p/45 + 2$$

Proof: We shall follow closely the proof of Theorem 10.4.4 of [**1**]. There an arc A is considered having $\#A = k$. Define $t = p + 2 - k$. Hirschfeld shows that the unisecants of A are contained in an envelope Δ_{2t} and he considers a component Γ_n of Δ_{2t}. He shows that if $t = 1$ then A is conic and for $t \geq 2$ there are three cases.

(i) Γ_n is a regular linear component

(ii) Γ_n is a regular component of class 2

(iii) Γ_n is regular of class at least three or Γ_n is irregular.

In case (i) he shows that A is not complete.

In case (ii) he shows that Γ_n is the envelope of a conic C, so by [**1**] Theorem 10.4.3 $A \subseteq C$ unless $k \leq (3p+5)/4 \leq 44p/45 + 2$.

In case (iii) if Γ_n has N simple lines, d double lines and $L = N + d$, he shows that $L \geq \frac{kn}{2}$.

By [**1**] Lemma 10.1.1, if Γ_n is irregular then $L \leq n^2$, so $k \leq 2n \leq 4t = 4(p+2-k)$. It follows that $k \leq \frac{4}{5}(p+2) \leq \frac{44}{45}p + 2$.

When Γ_n is regular we shall show below (Lemma) that (since $n \geq 3$) $N \leq \frac{1}{5}(10n(n-3)+2n(p+5)) - 4d$ if $t \leq p/4$. So when $t \leq p/4$ we have $\frac{1}{2}kn \leq \frac{1}{5}(10n(n-3)+2n(p+5))$ so $k \leq 4(n-3) + \frac{4}{5}(p+5) \leq 4(2t-3) + \frac{4}{5}(p+5) = 8t + \frac{4}{5}p - 8 = 8(p+2-k) + \frac{4}{5}p - 8$, so $k \leq \frac{44}{45}p + 2$. If $t > p/4$ then $k = p + 2 - t < \frac{3}{4}p + 2 \leq \frac{44}{45}p + 2$, this completes the proof.

The claim made in the proof follows immediately from the following.

LEMMA. *Let X be an absolutely irreducible plane curve of degree $n \geq 3$ over the field of p elements, where $p \neq 2$ is a prime number. If $n \leq p/2$, and the non-singular model of X has N rational points, then*

$$N \leq \frac{1}{5}(10n(n-3) + 2n(p+5)) - 4d,$$

where d is the number of double points of X as a plane curve.

Proof: It follows from the results of [**4**] (more precisely from Theorem 2.13 and Corollary 2.7) that if X is a curve defined over the field of q elements of characteristic p and has N rational points then if X is in projective r-space as a curve of degree $m \leq p$ and is not contained in any hyperplane then:

$$N \leq \frac{1}{r}(r(r-1)(g-1) + (q+r)m)$$

where g is the genus of X.

In our case we consider X embedded in $PG(5,p)$ by embedding $PG(2,p)$ in $PG(5,p)$ by the Veronese embedding (in projective coordinates $(x:y:z) \to (xy:xz:yz:x^2:y^2:z^2)$). The degree of X will be then $2n \leq p$, by hypothesis and X will not be contained in any hyperplane since $n \geq 3$. Then we can apply the above inequality with $m = 2n$, $r = 5$, $q = p$. The lemma will then follow upon noticing that

$$g \leq \frac{(n-1)(n-2)}{2} - d, \quad \text{so} \quad (g-1) \leq \frac{n(n-3)}{2} - d.$$

Remark: The result of this paper can be used to improve on several results on Finite Geometry in a fairly easy way (see, e.g. **[2]**, **[5]**, **[6]**).

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