Integral points of abelian varieties over function fields of characteristic zero

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This paper studies the question of integral points on affine open subsets of Abelian varieties over function fields of characteristic zero. Siegel [Sie] has shown that an affine algebraic curve of genus at least one defined over a number field, has only finitely many integral points. Lang [L] has proven an analogous result for curves defined over a function field of characteristic zero, not defined over the constant field. For higher dimensions, in the classical case of number fields, Lang has conjectured and Faltings [F] proved that for A an abelian variety over the number field K, if V is an affine open subset of A and S is a finite set of places of K, then the set of S-integral points of V is finite. Faltings has also a quantitative (but non-effective) result. Parshin [P] also obtained a result for function fields of characterisic zero, under the hypothesis that the complement of V does not contain translates of abelian subvarieties of A. In this paper, we remove this hypothesis and also obtain a quantitative result. Our main theorem is a boundedness result for the local height associated to a subvariety of an abelian variety defined over a discrete valuation field with residue characteristic zero; it is a higher dimensional analogue of a result of Manin [M] about elliptic curves. Our method is quite different from those in [M] and [P]; it is based on a differential algebraic argument from [B] plus an argument involving formal groups.

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Assume K is any field and $v : K_a \to (-\infty, \infty]$ is a real valuation on its algebraic closure. For any projective variety V over K and any closed subscheme $X \subset V$ we dispose of a function $\lambda_v(X,.) : V(K_a) \to [0,\infty]$ which satisfies the following property: for any affine open set $U \subset X$ and any system of generators $g_1,...,g_m \in \mathcal{O}(U)$ of the ideal defining $X \cap U$ in U, we may write $\lambda_v(X,P) = min\{v(g_1(P)),...,v(g_m(P))\}+b(P)$ with b bounded on any bounded subset of $U(K_a)$. The function $\lambda_v(X,.)$ is uniquely determined by the above property up to the addition of a bounded function and is called the *local height function* associated to X. This notion is developed in detail in [Sil].

Let us fix an algebraically closed ground field k of characteristic zero. All valuations are assumed to be trivial on k. Here are our main results:

Theorem. *Assume K is a field extension of k. Let A be an abelian variety over K with K/k trace zero. Let $X \subset A$ be a closed subvariety and let $\Gamma \subset A(K_a)$ be a finite rank subgroup. Let v be a real valuation on K_a which is discrete on K. Then $\lambda_v(X,P)$ is bounded for P in $\Gamma \setminus X$.*

Corollary. *Assume K is a function field over k. Let A be an abelian variety over K with K/k trace zero. Let $X \subset A$ be an ample divisor. Then for any finite set S of places of K, the set of S-integral points of $A \setminus X$ is finite.*

Proof of Corollary: The height $h(P)$, for a S-integral point of $A\setminus X$, is the sum of $\lambda_v(X,P)$ over the elements of S, and it follows that the height is bounded, which proves the corollary.

Remarks. 1) A few comments about terminology. We say that A has *K/k trace zero* if $A \otimes_K K_a$ contains no abelian subvariety defined over k. We say that Γ is of *finite rank* if $\Gamma \otimes_{\mathbf{Z}} \mathbf{Q}$ has finite dimension over $\mathbf{Q}$. So we do not assume that Γ in the Theorem is finitely generated; in particular if we let K in the Theorem be a function field over k, we may take Γ to be the group of division points of $A(K)$ i.e. the group of all points $P \in A(K_a)$ for which there exists a non-zero integer n_P with $n_P P \in A(K)$.

2) What the Theorem says is that there is no sequence in $\Gamma\setminus X$ converging $v-$adically to X (a sequence P_n in a projective variety is said to *converge $v-$adically* to X if $\lambda_v(X,P_n)$ goes to infinity as n goes to infinity.) It is worth recalling here that even if K in the Theorem is $v-$adically complete, $A(K)$ is not compact and a sequence in $A(K)$ converging $v-$adically to X may have no subsequence converging to a point of X.

To prove the Theorem we need some preparation. Let F be any field with a real valuation v.

1. We will need to talk about convergence of sequences of $F-$points of arbitrary (e.g. infinite dimensional) affine schemes. This is somewhat non-standard so we prefer to "recall" the definitions. Let $U = Spec\ R$ be an arbitrary affine $F-$scheme. A sequence $P_n \in U(F)$ will be called bounded (or bounded in U if there is any danger of confusion) if there exists a family (f_i) of $F-$algebra generators of $R = \mathcal{O}(U)$ such that for each i the sequence of real numbers $v(f_i(P_n))$ is bounded from below. If this holds then it is trivial to check that for any $f \in \mathcal{O}(U)$ the sequence $v(f(P_n))$ is bounded from below.

Let $X \subset U$ be a closed subscheme defined by an ideal I. A sequence $P_n \in U(F)$ is said to converge $v-$adically to X (in U, if there is any danger of confusion) if it is bounded and there exists a family (g_j) of generators of I with the property that for each j, $v(g_j(P_n))$ goes to infinity as n goes to infinity. If this holds then it is trivial to check that for any $g \in I$, $v(g(P_n))$ goes to infinity as n goes to infinity. It is also trivial to check that if P_n converges $v-$adically to X then it also converges $v-$adically to X_{red}.

2. Let U be an affine $F-$scheme and $X, Y \subset U$ closed subschemes. If a sequence $P_n \in U(F)$ coverges $v-$adically to both X and Y then it converges $v-$adically to $X \cap Y$.

Let $\pi : U \to U'$ be a morphism of affine $F-$schemes and let $X \subset U$ and $X' \subset U'$ be closed subschemes such that $\pi(X) \subset X'$ set-theoretically. Let $P_n \in U(F)$ be a sequence converging $v-$adically to X. Then $\pi(P_n)$ converges $v-$adically to X'.

3. Now we recall (and give some complements to) a construction done in [B]. Let δ be any derivation on F. Then for any $F-$scheme V there exists a pair $(jet_\infty(V), \tilde{\delta})$ where $jet_\infty(V)$ is a $V-$scheme and $\tilde{\delta}$ is a derivation on $\mathcal{O}_{jet_\infty(V)}$ prolonging δ, having the following universality property: for any pair (W, ∂) consisting of a $V-$scheme W and of a derivation ∂ of $\mathcal{O}_W$ which prolongs δ there exists a unique horizontal morphism of $V-$schemes $W \to jet_\infty(V)$. Here *horizontal* means commuting with the corresponding derivations. Note that $jet_\infty(V)$ is a "huge" scheme; e.g. it is always infinite dimensional for V an algebraic variety over F, of positive dimension. If $U \subset V$ is an open subscheme then $jet_\infty(U)$ naturally identifies with the inverse image of U in

$jet_\infty(V)$. If G is any algebraic $F-$group then $jet_\infty(G)$ has a natural structure of $F-$group scheme.

4. It is particularly useful to see how $jet_\infty(V)$ looks like in case

$$V = Spec\ F[y_1, ..., y_N]/(g_1, ..., g_m)$$

is an affine algebraic scheme. Indeed it follows from the universality property that we have

$$jet_\infty(V) = Spec\ F\{y_1, ..., y_N\}/[g_1, ..., g_m]$$

where $F\{y_1, ..., y_N\}$ is the "ring of $\delta-$polynomials" and $[g_1, ..., g_m]$ is the "$\delta-$ideal generated by $g_1, ..., g_m$". Recall that by definition the *ring of $\delta-$polynomials* is the usual ring of polynomials with coefficients in F in the infinite family of variables $(y_{ij}),\ 1 \leq i \leq N,\ j \in \mathbf{N}$ equipped with the unique derivation which prolongs δ and sends each y_{ij} into $y_{i,j+1}$ (call this unique derivation $\tilde{\delta}$). We always identify y_{i0} with y_i; in particular $g_1, ..., g_m$ are viewed as elements in the ring of $\delta-$polynomials. Now the "*$\delta-$ideal generated by $g_1, ..., g_m$*" is by definition the ideal generated by the infinite family $(\tilde{\delta}^j g_p),\ 1 \leq p \leq m,\ j \in \mathbf{N}$.

The description above shows that if $X \subset V$ is a closed subscheme of an $F-$variety then $jet_\infty(X)$ identifies with a closed horizontal subscheme of $jet_\infty(V)$; here *horizontal* means "whose ideal sheaf is preserved by $\tilde{\delta}$".

5. Let V be any $F-$scheme. Then by the universality property, any $F-$point P of V lifts to an $F-$point $jet_\infty(P)$ of $jet_\infty(V)$. Explicitly, if V is as in (4) and P is defined by $y_i \mapsto \alpha_i \in F$ then $jet_\infty(P)$ is defined by $y_{ij} \mapsto \delta^j \alpha_i \in F$. Assume U is an affine $F-$variety and let $f \in \mathcal{O}(U)$ be a regular function. Denote the image of f in $\mathcal{O}(jet_\infty(U))$ by the same letter f. We may consider the element $\tilde{\delta}^j f \in \mathcal{O}(jet_\infty(U))$ and evaluate it at the $F-$point $jet_\infty(P)$. On the other hand we may evaluate f at P and then take the $j-$th derivative in F. The description in (4) and the explicit form of $jet_\infty(P)$ given above show that what we obtain in both cases is the same: $(\tilde{\delta}^j f)(jet_\infty(P)) = \delta^j(f(P))$.

6. Assume now δ is a bounded derivation on F, by which we mean that $v(\delta x) \geq v(x)$ for all $x \in F$. Let U be an affine $F-$variety and $P_n \in U(F)$ a sequence of points.

We claim that if P_n is bounded in U then $jet_\infty(P_n)$ is bounded in $jet_\infty(U)$. Indeed by the description in (4), $\mathcal{O}(jet_\infty(U))$ is generated by elements of the form $\phi_{ij} := \tilde{\delta}^j f_i$ with $f_i \in \mathcal{O}(U)$ and $j \in \mathbf{N}$. By the formula at the end of (5) we get for each i and j:

$$v(\phi_{ij}(jet_\infty(P_n))) = v(\delta^j(f_i(P_n))) \geq v(f_i(P_n))$$

and our claim is proved.

We also claim that if $X \subset U$ is a closed subscheme and if P_n converges $v-$adically to X then $jet_\infty(P_n)$ converges $v-$adically to $jet_\infty(X)$. Indeed, by (4) again, the ideal defining $jet_\infty(X)$ in $jet_\infty(U)$ is generated by elements of the form $\psi_{ij} := \tilde{\delta}^j g_i$ with g_i in the ideal defining X and we conclude exactly as above.

The next result is an "approximation analogue" of Lang's conjecture on intersections of subvarieties of abelian varieties with finite rank subgroups. A similar result was also obtained by E. Hrushovski, (personal communication) using different methods.

Proposition 7. *Let F be an algebraically closed extension of k, v a real valuation on F, and δ a bounded derivation on F whose field of constants is k (i.e. $Ker\ \delta = k$). Let A be an abelian variety over F with F/k trace zero, let $X \subset A$ be a closed subvariety and $\Gamma \subset A(F)$ a finite rank subgroup. Then there exists in X a finite union Y of translates of abelian subvarieties with the property that any sequence P_n in Γ converging v-adically to X also converges $v-$adically to Y.*

Proof. Consider the scheme $jet_\infty(A)$, cf. (3) and let $\pi : jet_\infty(A) \to A$ be the canonical projection. Theorem 2 in [B] says that there is a horizontal, irreducible, closed $F-$subgroup scheme $H \subset jet_\infty(A)$ which is of finite type over F such that for any $P \in \Gamma$ we have $jet_\infty(P) \in H(F)$. So we dispose of two closed subschemes H and $jet_\infty(X)$ in $jet_\infty(A)$. Let Z be their scheme-theoretic intersection and let Y be the Zariski closure of $\pi(Z)$ in A. Then Theorem 1 in [B] says that any variety of general type dominated by a component of Y must have its Albanese variety descending to k. This plus our trace hypothesis easily implies (see the last page of [B] for the argument) that Y is a union of translates of abelian subvarieties. Now A can be covered by finitely many affine open sets U_i such that the sequence P_n is a union of subsequences P_{in}, each contained in the corresponding U_i and bounded in U_i. So to prove the Proposition we may assume there is an open affine subset $U \subset A$ such that $P_n \in \Gamma \cap U(F)$ converges $v-$adically to $X \cap U$ in U (in the sense of (1)) and we have to check that P_n also converges $v-$adically to $Y \cap U$ in U. Now by (6), $jet_\infty(P_n)$ converges $v-$adically to $jet_\infty(X \cap U) = jet_\infty(X) \cap jet_\infty(U)$ in $jet_\infty(U)$. On the other hand $jet_\infty(P_n)$ converges $v-$adically to $H \cap jet_\infty(U)$ (because it is bounded and contained in it). By (2), $jet_\infty(P_n)$ converges $v-$adically to $Z \cap jet_\infty(U)$. By (2) again, $\pi(jet_\infty(P_n)) = P_n$ converges $v-$adically to $Y \cap U$ and we are done.

Actually we proved more. Assume we are in the hypotheses of the Proposition. Let $\Gamma^* \subset A(F)$ be the $\delta-$closure of Γ (cf. [B], p.560 for the definition of "$\delta-$closure".) Then we actually proved that any sequence P_n of points in Γ^* which converges $v-$adically to X also converges $v-$adically to Y. This might have some interest in its own because the rank of Γ^* is generally infinite.

Proposition 8. *Let L be a complete real valued field with residue characteristic zero. Let G be a commutative analytic group over L. Let $\Gamma \subset G$ be a finite rank subgroup. Then Γ is discrete in the $v-$adic topology of G.*

Proof. We refer to [Se], Chapter 4, for background. For any real $\alpha \geq 0$ let I_α be the additive group of all elements of L whose valuation is $\geq \alpha$. Since G contains an open subgroup which is standard (in the sense of [Se], i.e. it is the group of points of a formal group over the valuation ring of L) we may assume G itself is standard. Then G has a filtration $(G_i)_{i \in \mathbf{N}}$ with open subgroups such that G_i/G_{i+1} is isomorphic to the group $(I_i/I_{i+1})^g$, for all $i \in \mathbf{N}$, where g is the dimension of G, and $\bigcap_{i \in \mathbf{N}} G_i = \{0\}$. Assume there exists a sequence P_i in $\Gamma \backslash \{0\}$ converging to 0. Then we may assume there exists a sequence of integers $0 < k_1 < k_2 < ...$ such that $P_i \in G_{k_i} \backslash G_{k_i+1}$. Then we claim that P_i are $\mathbf{Z}-$linearly independent in G; this will be a contradiction, and will close our proof. To check the claim assume $n_i P_i = \sum_{j>i} n_j P_j$, $n_i \neq 0$. Then $n_i P_i \in G_{k_{i+1}} \subset G_{k_i+1}$. Since $(I_{k_i}/I_{k_i+1})^g$ is torsion free, it follows that $P_i \in G_{k_i+1}$, a contradiction, and we are done.

9. *Proof of the Theorem.* Embed K into its $v-$adic completion $k_1((t))$, where k_1 is the residue field, t is some variable and v on K is the restriction of the valuation $v_t =$"order of series in t". There exists a $K-$embedding of valued fields $(K_a, v) \subset (k_{1a}((t))_a, v_t)$; note that $k_{1a}((t))_a = \bigcup_{q \in \mathbf{N}} k_{1a}((t^{1/q}))$. Now countably many series in $k_{1a}((t))_a$ are enough to define our data A, X, Γ so

these data are defined over $k_2((t))_a$, where k_2 is some countably generated extension of k contained in k_{1a}. We may embed k_2 over k into $k((s))_a = \bigcup_{p \in \mathbf{N}} k((s^{1/p}))$ where s is a new variable. We get an embedding $k_2((t))_a \subset F := k((s))_a((t))_a$. Then F is an algebraically closed real valued field (with valuation v_t). Consider on F the bounded derivation $\delta := s^2\partial_s + t\partial_t$ where $\partial_s := \partial/\partial s,\ \partial_t := \partial/\partial t$. We claim that $Ker\ \delta = k$. Indeed if $\delta f = 0$ for some series $f = \sum f_n t^{n/q} \in F,\quad f_n \in k((s^{1/p_n}))$, then for any n such that $f_n \neq 0$ we get $s^2(f_n^{-1}\partial_s f_n) = -n/q$. Let $v_s : k((s))_a^* \to \mathbf{Q}$ be the valuation defined by "order of series in s". Since $v_s(f_n^{-1}\partial_s f_n) \geq -1$, we must have $n = 0$. So we must have $f = f_0$ and $\partial f_0/\partial s = 0$, hence $f \in k$ and our claim is proved.

Take now any sequence P_n in $\Gamma \backslash X$. We claim that P_n cannot converge $v-$adically to X, and this will close the proof of the Theorem. Assume it does. Then the same will hold over $k_{1a}((t))_a$, hence over $k_2((t))_a$, hence over F. By Chow's rigidity theorem the abelian variety A_F over F corresponding to A will still have F/k trace zero. By Proposition 7, P_n converges v_t-adically to a finite union of translates of abelian subvarieties of A_F. Passing to a subsequence we may assume P_n converges v_t-adically to a translate of an abelian subvariety $B \subset A_F$. Let $\pi : A_F \to C := A_F/B$ be the canonical projection. Then $\pi(P_n)$ converges v_t-adically to a point of C. Let L be the completion of F. Then the group $C(L)$ is an analytic group over L and $\pi(\Gamma)$ is not discrete in it, this contradicting Proposition 8. Our Theorem is proved.

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