[12pt]article 16cm 21.5cm -1.5cm -1mm -1mm 3mm 3mm 0mm 2mm 12pt 12pt 1.1in Lang's conjecture in characteristic p: an explicit bound Alexandru Buium and José Felipe Voloch document

Let K be a function field in one variable with constant field k and denote by K_a, K_s its algebraic and separable closures, respectively. Let X/K be an algebraic curve of genus at least two. The function field analogue of Mordell's conjecture states that $X(K)$ is finite unless X is K_a- isomorphic to a curve defined over k, in which case X is called isotrivial. This was first proved by Manin [Man] in characteristic zero and, shortly after, another proof was given by Grauert [Gra] and this proof was then adapted by Samuel [Sa] to positive characteristic. Since then several different proofs were given for Mordell's conjecture over function fields. In particular Szpiro [Sz] was the first to prove an effective version of Mordel's conjecture in characteristic p.

Mordell's conjecture was generalized by Lang [L] (and proved through work of Raynaud [R] and Faltings [F]). An analogue of Lang's conjecture over function fields of characteristic p was proved by the second author and Abramovich [V] [AV]. The aim of the present paper is to prove an effective version of Lang's conjecture in characteristic p. Our approach consists of combining the approaches in [B1] and [V] [AV] which in their turn were motivated by Manin's work [Man]. Here is our main result:

Theorem. *Let K be a function field in one variable and characteristic $p > 0$. Let X be a smooth projective curve of genus $g \geq 2$ over K embedded into its Jacobian J. Assume X has non-zero Kodaira-Spencer class (equivalently, X is not defined over K^p). If Γ is a subgroup of $J(K_s)$ such that $\Gamma/p\Gamma$ is finite, then:* $\sharp(X \cap \Gamma) \leq \sharp(\Gamma/p\Gamma) \cdot p^g \cdot 3^g \cdot (8g-2) \cdot g!$

We stress the fact that we do not assume Γ is finitely generated, which is the main feature in Lang's conjecture that distinguishes it from the Mordell conjecture .

A similar result in characteristic zero was obtained by the first author [B4] but the bound there is huge as compared to the bound here. This is a reflection of the fact that the characteristic p case is in some sense "easier" than the characteristic zero case.

The question of the existence of this type of bounds for Lang's conjecture was raised by Mazur in [Maz] and is quite different from what one understands by "effective Mordell". In particular, even in the special case when Γ in our Theorem is finitely generated, our bound is not a consequence of Szpiro's [Sz]. Indeed, assuming we are in the hypothesis of the Theorem above with Γ finitely generated, let $K_1 \subset K_s$ be the field generated over K by the coordinates of the points in Γ. What Szpiro's "effective Mordell" yields is a bound for the height of the points in $X(K_1)$ that depends on g =genus of X, p =characteristic of k, q_1 =genus of K_1 and s_1 =number of points of bad reduction of a semistable model of $X \otimes K_1/K_1$. It follows that $\sharp(X \cap \Gamma)$ is bounded by a constant that depends on g, p, q_1, s_1. But of course q_1, s_1 are not bounded by a constant that depends on $\sharp(\Gamma/p\Gamma)$ only; we may always keep $\sharp(\Gamma/p\Gamma)$ constant and vary Γ so that both q_1 and s_1 go to infinity.

In order to prove the Theorem let us start by recalling a construction from [B1]. Assume we have fixed a derivation $\delta = \partial/\partial t$ of K where $t \in K$ is a separable transcendence basis of K/k. Then for any $K-$scheme X one defines the "first jet scheme along δ" by the formula X$^1 := Spec(S(\Omega_{X/k})/I) where$I$istheidealgeneratedbysectionsofth$ δf $(f \in \mathcal{O}_X)$. This object was analysed in [B2], [B3] where the characteristic zero case only was considered. But many of the facts proved there extend, with identical proofs, in positive characteristic. In particular the following hold. Assume X above is a smooth variety over K. Then exactly as in [B1], p.1396, X^1 identifies with the torsor for the tangent bundle $TX := Spec(S(\Omega_{X/K}))$ corresponding to the Kodaira Spencer class $\rho(\delta) \in H^1(X, T_{X/K})(where \rho : Der_k K \to H^1(X, T_{X/K})$ is the Kodaira Spencer map; this map played various roles in virtually all approaches to the Mordell and Lang conjectures over function fields.) So exactly as in [B2], section 1, we may write X^1 as the complement of a divisor in a projective bundle: X$^1 = \mathbf{P}(E) \backslash \mathbf{P}(\Omega_{X/K}) where$E$isthevectorbundledefinedbytheextension 0 \to \mathcal{O}_X \to E \to \Omega_{X/K} \to 0 corresponding to \rho(\delta) \in H^1(X, T_{X/K}) \simeq Ext^1(\Omega_{X/K}, \mathcal{O}_X)$.

If X/K is a smooth group scheme then so is X^1/K.

Also, since δ lifts to a derivation of K_s, there is an obvious "lifting map" $\nabla : X(K_s) \to X^1(K_s) whichincaseX/Kisagro$

The following is the characteristic p analogue of a fact from [B3], (2.2):

Lemma. *If X/K is a smooth projective curve of genus ≥ 2 with non zero Kodaira Spencer class then X^1 is an affine surface.*

Proof. By the discussion preceding the Lemma it is enough to check that the divisor $\mathbf{P}(\Omega_{X/K})$ is ample in $\mathbf{P}(E)$, equivalently that E is ample, which is the same as E_a being ample (where E_a is the pull back of E on $X_a := X \otimes_K K_a$). Let $F : X_a \to X_a$ be the absolute Frobenius (viewed as a scheme morphism over the integers). Assume E_a is not ample and seek a contradiction. By the characteristic p analogue of "Gieseker's Theorem" [Gie] due to Martin-Deschamps [MD] it follows that there exists a power $F^m : X_a \to X_a$ of F such that the pull back of the sequence (*) $\quad 0 \to \mathcal{O}_{X_a} \to E \to \Omega_{X_a/K_a} \to 0 via F^m$ splits. Now, since Ω_{X_a/K_a} has degree $2g-2 > (2g-2)/p$, a result of Tango [T] Theorem 15 p. 73 implies that the sequence (*) itself must be split, which contradicts the fact that the Kodaira-Spencer class of X/K is non zero. This completes the proof of the Lemma.

Proof of the Theorem. The closed immersion $X \subset J$ induces a closed immersion $X^1 \subset J^1$. For any point $P \in X(K_s) \cap pJ(K_s)$ we have $\nabla(P) \in X^1(K_s) \cap pJ^1(K_s)$. Since J^1 is an extension of J by a vector group (same argument as in [B2] (2.2)) the algebraic group $B = pJ^1$ coincides with the maximum abelian subvariety of J^1 and the projection $B \to J$ is an isogeny. Moreover by [Ro], p.704, Lemma 2, the natural isogeny (the Verschiebung) $J^{(p)} \to J$ factors through $B \to J$. Since Verschiebung is of degree p^g, $B \to J$ has degree at most p^g.

In order to prove the Theorem it is obviously enough to prove that, over K_a, $X^1 \cap B$ is finite, of cardinality at most $p^g \cdot 3^g \cdot (8g-2) \cdot g!$. Finiteness follows trivially from our Lemma above: X^1 is affine and B is complete and both are closed in J^1 so their intersection is closed in both X^1 and B, so $X^1 \cap B$ is both affine and complete, hence it is finite over K. To estimate its cardinality we use Bézout's theorem in Fulton's form, along the lines of [B4] (except that here we do not need any "iteration" and we do not have to take multiplicities into account !).

Recall that X^1 and J^1 are Zariski locally trivial principally homogeneous spaces for the tangent bundles of X and J respectively. Let $\eta_X \in H^1(X, T_{X/K})$ and $\eta_G \in H^1(J, T_{J/K})$ be the corresponding cohomology classes defining these homogeneous spaces and let $0 \to \mathcal{O}_X \to E_X \to \Omega_{X/K} \to 0 0 \to \mathcal{O}_J \to E_J \to \Omega_{J/K} \to 0 be the extension corresponding to \eta_X, \eta_J$ respectively. Consider the divisors $D_X = \mathbf{P}(\Omega_{X/K}) \subset \mathbf{P}(E_X)$ and $D_J = \mathbf{P}(\Omega_{J/K}) \subset \mathbf{P}(E_J)$. Since $\Omega_{J/K} \simeq \mathcal{O}_J^g$ we have $D_J \simeq J \times \mathbf{P}^{g-1}$. Recall also that these divisors belong to the linear systems associated to $\mathcal{O}_{\mathbf{P}(E_X)}(1)$ and $\mathcal{O}_{\mathbf{P}(E_J)}(1)$ respectively and that we have identifications $X^1 \simeq \mathbf{P}(E_X) \backslash D_X$ and $J^1 \simeq \mathbf{P}(E_J) \backslash D_J$. Let $\alpha : X \to J$ be the inclusion. We claim there is a natural restriction homomorphism $\alpha^* E_J \to E_X$ prolonging the natural homomorphism $\alpha^* \Omega_{J/K} \to \Omega_{X/K}$. Indeed E_X, E_J are subsheaves of the direct image sheaves $\pi_{X*}\mathcal{O}_{X^1}$ and $\pi_{J*}\mathcal{O}_{J^1}$ where $\pi_X : X^1 \to X$ and $\pi_J : J^1 \to J$ are the natural projections. These direct image sheaves have natural filtrations induced by the $\mathbf{G}_m$−action on the tangent bundles, and E_X, E_J identify with the first piece of this filtration. Now we have a natural map $\alpha^* \pi_{J*}\mathcal{O}_{J^1} \to \pi_{X*}\mathcal{O}_{X^1}$. This map is compatible with the $\mathbf{G}_m$−actions in an obvious way so it preserves filtrations; in particular it sends $\alpha^* E_X$ into E_J. (Cf. [B3], section 1, for details in an analogous situation.) The homomorphism $\alpha^* E_J \to E_X$ is clearly surjective so it induces a closed embedding $\mathbf{P}(E_X) \subset \mathbf{P}(E_J)$ prolonging the embedding $X^1 \subset J^1$. By abuse we shall still denote by π_X, π_J the projections $\mathbf{P}(E_X) \to X, \mathbf{P}(E_J) \to J$.

Claim. The line bundle $\mathcal{H} := \pi_J^* \mathcal{O}_J(3\Theta) \otimes \mathcal{O}_{\mathbf{P}(E_J)}(1)$ is very ample on $\mathbf{P}(E_J)$. (Here Θ is the theta divisor on J.)

To check the Claim, note first that the trace of the linear system $|\mathcal{H}|$ on D_J is very ample. Indeed $\mathrm{H} \otimes \mathcal{O}_{D_J} = \mathcal{H} \otimes \mathcal{O}_{\mathbf{P}(\Omega_{J/K})} = p_1^* \mathcal{O}_J(3\Theta) \otimes p_2^* \mathcal{O}_{\mathbf{P}^{g-1}}(1) where \mathrm{p}_1, p_2$ are the two projections of $D_J = J \times \mathbf{P}^{g-1}$ onto the factors. So $\mathcal{H} \otimes \mathcal{O}_{D_J}$ is very ample on D_J, cf. [Mum] p. 163. Furthermore we have an exact sequence $\mathrm{H}^0(\mathbf{P}(E_J), \mathcal{H}) \to H^0(D_J, \mathcal{H} \otimes \mathcal{O}_{D_J}) \to H^1(\mathbf{P}(E_J), \pi_J^* \mathcal{O}_J(3\Theta)) But the \mathrm{H}^1$ above is zero (use the Leray spectral sequence and the vanishing theorem in [Mum] p.150) so the trace of $|\mathcal{H}|$ on D_J is a complete linear system and hence is very ample. In particular $|\mathcal{H}|$ separates points of D_J and "vectors tangent to D_J". Since $|\mathcal{H}|$ has no base points outside D_J either, it follows that $|\mathcal{H}|$ is base point free on $\mathbf{P}(E_J)$. Hence $|\mathcal{H}|$ restricted to the fibres of π_J is base point free. Since any base point free linear subsystem of $|\mathcal{O}_{\mathbf{P}^g}(1)|$ equals actually the whole of $|\mathcal{O}_{\mathbf{P}^g}(1)|$ it follows that $|\mathcal{H}|$ separates points in each fibre of π_J and separates "vectors tangent to each fibre". All these imply that $|\mathcal{H}|$ separate points and tangent vectors on the whole of $\mathbf{P}(E_J)$ and our Claim is proved.

Our last step is to compute the degrees $deg_{\mathcal{H}}\mathbf{P}(E_X)$ and $deg_{\mathcal{H}}B$ of $\mathbf{P}(E_X)$ and B respectively, as subvarieties of $\mathbf{P}(E_J)$ with respect to the embedding defined by $\mathcal{H}$. Note that H$\otimes\mathcal{O}_{\mathbf{P}(E_X)} = \pi_X^*\mathcal{O}_X(3\Theta) \otimes \mathcal{O}_{\mathbf{P}(E_X)}(1) Wemaycomputetheselfintersection (\mathcal{O}_{\mathbf{P}(E_X)}(1)\cdot\mathcal{O}_{\mathbf{P}(E_X)}(1))_{\mathbf{P}(E_X)} = deg\ \Omega_{X/K} = 2g-2andsince(\Theta\cdot X)_J = g$ we get $\deg_{\mathcal{H}}\mathbf{P}(E_X) = 2g-2+6g = 8g-2 Ontheotherhandwehave \mathcal{H}\otimes\mathcal{O}_B \simeq \pi^*\mathcal{O}_J(3\Theta) where \pi :$ $B \to J$ is the projection which we already know has degree at most p^g. So we get, using $(\Theta^g)_J = g!$, that $\deg_{\mathcal{H}}B = p^g\cdot 3^g\cdot g! NowBezout'stheoreminFulton'sform[Fu]p.148, saysthatthenumberofirreduciblecomponentsinthein$ cannot exceed $d_1 d_2$. In particular $\sharp(X^1\cap B) \leq deg_{\mathcal{H}}\mathbf{P}(E_X)\cdot deg_{\mathcal{H}}B \leq (8g-2)\cdot p^g\cdot 3^g\cdot g! andourTheoremisproved.$

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