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Reduction of the Manin map modulo p Alexandru Buium and José Felipe Voloch document

For an abelian variety A over a function field K of characteristic zero equipped with a derivation $\delta : K \to K$ Manin defined in [Man1], [Man2] a remarkable additive map $A(K) \to V$, where V is a vector space over K, which plays an important role in diophantine geometry over function fields. (Cf. [Co] for a "modern" exposition of Manin's work. Cf. also [B1], [B2] for a different way of introducing this map.) In the "generic case" this map is a "second order non linear differential operator" and the dimension of V is the dimension of A. In [V1] the second author defined an analogue of this map in the case of elliptic curves over function fields of characteristic p. This analogous map turned out to be of order one. In the higher dimensional case one also has a "characteristic p analogue of the Manin map", which is implicit in [BV] . (Cf. also [H] for a different approach.) We shall give the definition of this map below. In [V1] it was proved, for ordinary elliptic curves, that the "reduction mod p" of the Manin map in characteristic zero is the "derivative" of the analogue of the Manin map in characteristic p and that the kernel of the analogue of the Manin map in characteristic p is the group of points divisible by p. In this paper we will generalise these results to abelian varieties of arbitrary dimension.

The method used in [V1] to prove the first result mentioned above was based on a computation with Tate curves hence seems to be hard to extend to the higher dimensional case; instead we use the approach in [B1], [B2]. Note also that since this result is a statement about algebraic groups (in characteristic $p > 0$), it makes sense to consider the statement which it implies at the level of Lie algebras; it turns out that the corresponding Lie algebra statement is (a dual form of) a basic result of Igusa-Manin-Katz [Man3], [K1] which says, roughly speaking, that the Hasse-Witt matrix satisfies the Picard-Fuchs equation. But of course there is no way back, in characteristic $p > 0$, from Lie algebras to algebraic groups; so there is no way back from the Igusa-Manin-Katz result to our result.

On the other hand, we shall prove the generalization of the second result by extending the approach of [V1] by relating the analogue of the Manin map in characteristic p to the p-descent map. Note also that this result is an analogue in characteristic p of the Manin-Chai "Theorem of the kernel" [Man1] [Man2], [Ch].

To state our main result let us recall first some general definitions from [B1,2] which lead to a general concise definition of the "Manin maps" in arbitrary characteristic. Let A be a scheme over a scheme S and assume we are given a derivation δ on the structure sheaf of S. Then one can form a projective system of $S-$schemes (A^n) for $n \geq -1$ with affine transition maps π_n and derivations $\delta = \delta_n : \mathcal{O}_{A^n} \to \pi_{n*}\mathcal{O}_{A^{n+1}}$ as follows. Set $A^{-1} = S$, $A^0 = A$; let δ_{-1} be induced by δ and define inductively $A^{n+1} = \text{Spec } (S(\Omega_{A^n})/I_n)$ where I_n is the ideal generated by sections of the form $d(\pi_{n-1}^* f) - \delta_{n-1} f,\ f \in \mathcal{O}_{A^{n-1}}$, while δ_n is induced by the Kahler differential d in the obvious way. The schemes A^n are called the *schemes of $n-$jets of A/S along the direction* δ, or simply the *canonical prolongations of A/S*. This construction commutes, in the obvious sense, with "horizontal" base change $(S', \delta') \to (S, \delta)$ (here "horizontal" means "compatible with the derivations") and has the following universality property: for any A^n-scheme Z and any derivation ∂ from $\mathcal{O}_{A^n}$ to (the direct image of) $\mathcal{O}_Z$ prolonging the derivation δ_{n-1} there exists a unique A^n-scheme morphism $Z \to A^{n+1}$ such that ∂ is induced by δ_n. We refer to [B2] Part I for details.

Assume now in addition that A/S is a smooth group scheme. Then (A^n) will be a projective system of smooth group schemes. Denote by $\mathbf{X}^n(A)$ the set of all $S-$group scheme homomorphisms from A^n to the additive group $\mathbf{G}_a$ over S. This set has actually a structure of $\mathcal{O}(S)-$submodule of the ring $\mathcal{O}(A^n)$. Moreover the maps $\mathcal{O}(A^n) \to \mathcal{O}(A^{n+1})$ induced by the π_n's are injective and will be viewed as inclusions; so we get induced derivations $\delta : \mathcal{O}(A^n) \to \mathcal{O}(A^{n+1})$ which induce maps $\delta : \mathbf{X}^n(A) \to \mathbf{X}^{n+1}(A)$. The space $\mathbf{X}^n(A)$ was called in [B2] the *space of $\delta-$polynomial characters of A of order* $\leq n$ (See [B2], Part I, §6 or Part II, Introduction. There it was denoted by $\mathbf{X}_a^{[n]}(A)$.) Note that each element ψ of $\mathbf{X}^n(A)$ defines a homomorphism $\hat{\psi} : A(S) \to \mathcal{O}(S)$ by the formula $\hat{\psi}(P) = \psi_S(\nabla_n(P)), \quad P \in A(S) where \psi_S : A^n(S) \to \mathbf{G}_a(S) = \mathcal{O}(S)$ is the map induced by ψ and $\nabla_n(P) \in A^n(S)$ stands for the "canonical lifting" of P ([B2], Part I, (3.8)). These homomorphisms $\hat{\psi}$ are what we call *Manin maps.* The components of the classical Manin map in characteristic zero [Man1] [Man2] as well as the Manin map in characteristic p in [V1] are all of the form $\hat{\psi}$ above; so studying Manin maps is the same as studying the spaces $\mathbf{X}^n(A)$.

Throughout the paper we shall consider the following situation. We start with a discrete valuation ring

R and we denote by K and $\bar{K}$ the quotient field and the residue field of R respectively. We assume K has characteristic zero and $\bar{K}$ is a function field of one variable over a perfect field of characteristic $p > 0$. As a general rule the upper bar will denote the reduction modulo the maximal ideal m_R of R; in particular for any element $a \in R$ we denote its image in $\bar{K}$ by $\bar{a}$. We assume we are given a derivation $\delta : R \to R$ such that $\delta(m_R) \subset m_R$ (this is automatic if p is tamely ramified in R) and such that $\delta(R) \not\subset m_R$. Then this derivation will induce a non zero derivation (still denoted by) δ on $\bar{K}$. (A typical example of this situation is: $R = \mathbf{Z}[t]_{(p)}$, $\delta = \partial/\partial t$, $K = \mathbf{Q}(t)$, $\bar{K} = \mathbf{F}_p(t)$.)

Next we consider an abelian scheme A/R of relative dimension g; let A_K/K and $\bar{A}/\bar{K}$ be the generic and special fibres respectively and consider the corresponding spaces of $\delta-$polynomial characters $\mathbf{X}^n(A), \mathbf{X}^n(A_K), \mathbf{X}^n(\bar{A})$. We will make the following two assumptions:

(i) $\bar{A}$ has $p-$rank g (one also says that $\bar{A}$ is *ordinary*). Recall that this means that the rank of the Frobenius endomorphism of $H^1(\bar{A}, \mathcal{O})$ equals g (recall that the rank of a $p-$linear map is the dimension of the linear span of its image).

(ii) $\bar{A}$ has $\delta-$rank g (cf. [B2], Part I, (6.5)). Recall that this means that the $\bar{K}-$linear map $\rho(\delta)\cup : H^0(\bar{A}, \Omega^1) \to H^1(\bar{A}, \mathcal{O})$ induced by cup product with the Kodaira Spencer class $\rho(\delta) \in H^1(\bar{A}, T_A)$ has rank g, where T_A is the tangent bundle of A and $\rho : Der\ \bar{K} \to H^1(\bar{A}, T_A)$ is the Kodaira-Spencer map.

Of course (ii) implies that A_K itself has $\delta-$rank g (in the analogous sense). So by [B2], Part I, Proposition (6.6), $\mathbf{X}^1(A_K) = 0$ and that $\mathbf{X}^2(A_K)$ has dimension g over K. The "classical Manin map in characteristic zero" in this case is the map $(\hat{\psi}_1, ..., \hat{\psi}_g) : A(K) \to K^g$ associated to a basis $\psi_1, ..., \psi_g$ of $\mathbf{X}^2(A_K)$.

We have a natural inclusion $\mathbf{X}^2(A) \subset \mathbf{X}^2(A_K)$; elements of $\mathbf{X}^2(A_K)$ which lie in $\mathbf{X}^2(A)$ will be called *integral*. We also have a reduction modulo m_R map $\mathbf{X}^2(A) \to \mathbf{X}^2(\bar{A}), \quad \psi \mapsto \bar{\psi} sotheintegralelements \psi$ of $\mathbf{X}^2(A_K)$ may be reduced modulo m_R to get elements $\bar{\psi} \in \mathbf{X}^2(\bar{A})$. Finally recall that we have have a map induced by derivation $\mathbf{X}^1(\bar{A}) \to \mathbf{X}^2(\bar{A}), \quad \phi \mapsto \delta\phi Ourmainresultisthefollowing :$

Theorem. *There exists an integral basis $\psi_1, ..., \psi_g$ of $\mathbf{X}^2(A_K)$ and elements $\phi_1, ..., \phi_g \in \mathbf{X}^1(\bar{A})$ with the following properties:*

Actually, as we shall recall below, there is a natural increasing "filtration by degrees" $F^d\mathbf{X}^1(\bar{A}), d \geq 0$ on $\mathbf{X}^1(\bar{A})$ (cf. [B2], Part II, Introduction); then we shall prove that $F^{p-1}\mathbf{X}^1(\bar{A}) = 0$ and that $\phi_1, ..., \phi_g$ in the Theorem may be taken to be a basis of $F^p\mathbf{X}^1(\bar{A})$.

In what follows we devote ourselves to the proof of the Theorem. In the end of the paper we will show how to deduce from our Theorem the (dual form of the) Igusa-Manin-Katz theorem by passing to Lie algebras.

We need a "cocycle description" of the various objects involved; cf. [B3]. Let $\{U_i\}$ be a finite affine open cover of A and let θ_i be derivations of $\mathcal{O}(U_i)$ which lift the derivation δ of R. Fix a basis $\omega_1, ..., \omega_g$ of the $R-$module $H^0(A, \Omega^1)$ and let $v_1, ..., v_g$ be the dual basis of $L(A)$ =Lie algebra of A/R. We view elements of $L(A)$ as derivations of $\mathcal{O}_A$. Then we may write (1) $\quad \theta_j - \theta_i = \sum_{n=1}^{g} a_{ijn} v_n with a_{ijn} \in \mathcal{O}(U_i \cap U_j)$. Then the a_{ijn} are cocycles; let $e_n \in H^1(A, \mathcal{O})$ be the classes of these cocycles. Since the reduction modulo m_R of the cocycle $\theta_j - \theta_i$ represents the Kodaira Spencer class $\rho(\delta) \in H^1(\bar{A}, T)$ it follows that the images $\bar{e}_1, ..., \bar{e}_g \in H^1(\bar{A}, \mathcal{O})$ of the e_i's are a basis of $H^1(\bar{A}, \mathcal{O})$ (image of $\bar{\omega}_1, ..., \bar{\omega}_g \in H^0(\bar{A}, \Omega^1)$ via the map $\rho(\delta) : H^0(\bar{A}, \Omega^1) \to H^1(\bar{A}, \mathcal{O})$) hence $e_1, ..., e_g$ form a basis of $H^1(A, \mathcal{O})$.

Now our choice of a basis for the tangent bundle of U_i and of the lifting θ_i provides an U_i-isomorphism $\sigma_i : U_i^1 \to U_i \times$ Spec $R[x_1, ..., x_g] such that the derivation \delta : \mathcal{O}(U_i) \to \mathcal{O}(U_i^1)$ corresponds to the derivation (2) $\quad \delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i)[x_1, ..., x_g], \quad \delta_i = \theta_i + \sum_{n=1}^{g} x_n v_n It follows from [B2], Part I, (2.4) that$ A^1 is the universal vectorial extension of A by $\mathbf{G}_a^g$ (actually in [B2] one assumes a ground field rather than a ground discrete valuation ring, but all arguments go through). Then the isomorphisms σ_i are $\mathbf{G}_a^g-$equivariant where $\mathbf{G}_a^g$ acts on the affine space Spec $R[x_1, ..., x_g]$ by translations on the affine coordinates. Setting $x_{in} := \sigma_i^* x_n$

we get from (1) and (2) that (3) $\quad a_{ijn} = x_{in} - x_{jn} in the ring O(U^1_{ij})$, $U_{ij} := U_i \cap U_j$. Applying δ we get (4) $\quad \delta a_{ijn} = \delta x_{in} - \delta x_{jn} in the ring O(U^2_{ij})$.

Reducing the isomorphisms σ_i modulo m_R we get isomorphisms $\bar{\sigma}_i : \bar{U}^1_i \to \bar{U}_i \times \text{Spec } \bar{K}[\bar{x}_1, ..., \bar{x}_g] The pullbacks of \bar{x}_1, ..$ via $\bar{\sigma}_i$ restricted to $L(\bar{A}) = \ker(\bar{A}^1 \to \bar{A})$ will be affine maps whose associated linear maps are $\bar{\omega}_1, ..., \bar{\omega}_g$. Now the rings $\mathcal{O}(\bar{U}_i)[\bar{x}_1, ..., \bar{x}_g]$ have a natural filtration by degrees (the $d-$piece of the filtration being the space of all polynomials of degree $\leq d$). These filtrations induce via $\bar{\sigma}_i$ filtrations on $\mathcal{O}(\bar{U}^1_i)$ (the $d-$th piece of this filtration is the space of all $\bar{K}-$linear combinations of $e-$fold products of elements $\bar{x}_{i1}, ..., \bar{x}_{ig}$ with $e \leq d$.) Clearly these filtrations glue together to give a filtration $F^d \mathcal{O}(\bar{A}^1)$ on $\mathcal{O}(\bar{A}^1)$ and hence an induced filtration $F^d \mathbf{X}^1(\bar{A})$ on $\mathbf{X}^1(\bar{A})$. We could introduce similar filtrations on $\mathbf{X}^1(A), \mathbf{X}^1(A_K)$ but we won't need them in what follows. Here are two basic facts about the above introduced filtration.

Lemma 1. $F^{p-1}\mathbf{X}^1(\bar{A}) = 0$.

Proof. Let $\phi \in F^{p-1}\mathbf{X}^1(\bar{A})$. Then the restrictions of ϕ to $\bar{U}^1_i$ have the form $\bar{\sigma}^*_i H_i$ where $H_i = H_i(\bar{x}_1, ..., \bar{x}_g)$ are polynomials of degree $\leq p-1$ with coefficients in $\mathcal{O}(\bar{U}_i)$. By the $\mathbf{G}^g_a-$equivariance of σ_i, H_i must have the form $H_i = h_i + \sum_{m \geq 0} \sum_{n=1}^g h_{inm} \bar{x}_n^{p^m} where h_i, h_{inm} \in \mathcal{O}(\bar{U}_i)$. Since we are looking at the $F^{p-1}-$piece of the filtration we must have $h_{inm} = 0$ for $m \geq 1$. The conditions $(\bar{\sigma}^*_j)^{-1} \bar{\sigma}^*_i H_i = H_j$ give (5) $\quad h_i + \sum_{n=1}^g h_{in0}(\bar{x}_n + \bar{a}_{ijn}) = h_j + \sum_{n=1}^g h_{jn0} \bar{x}_n This implies that (h_{in0})_i$ glue together to give an element $k_n \in \mathcal{O}(\bar{A}) = \bar{K}$. Then (5) further implies that $\sum_{n=1}^g k_n \bar{a}_{ijn} = h_j - h_i Passing to cohomology classes one gets \sum_{n=1}^g k_n \bar{e}_n = 0$ in $H^1(\bar{A}, \mathcal{O})$ so we get $k_n = 0$ for all n. So $H_i = h_i$ for all i. Again the h_i glue together to give an element in $\bar{K}$ which must be 0 hence $\phi = 0$ and we are done.

Lemma 2. $\dim F^p \mathbf{X}^1(\bar{A}) = g$.

Proof. Firstly, we construct g linearly independent elements $\phi_1, ..., \phi_g$ in $F^p\mathbf{X}^1(\bar{A})$; this construction will be used later. By the $p-$rank condition we may write $\bar{e}_n^{(p)} = \sum_{m=1}^g \bar{\lambda}_{nm} \bar{e}_m, \quad 1 \leq n \leq g \lambda_{nm} \in R$, $det(\bar{\lambda}_{nm}) \neq 0$, where the upper (p) means "image under the Frobenius". The matrix $\bar{\lambda} = (\bar{\lambda}_{nm})$ is classically called the *Hasse-Witt matrix* (corresponding to the basis $\bar{e}_1, ..., \bar{e}_g$). So we may write (6) $\bar{a}^p_{ijn} - \sum_{m=1}^g \bar{\lambda}_{nm} \bar{a}_{ijm} = \bar{a}_{in} - \bar{a}_{jn}, \ \ 1 \leq n \leq g where a_{in} \in \mathcal{O}(U_i)$. Define (7) $\quad \phi_{in} := \bar{x}^p_{in} - \sum_{m=1}^g \bar{\lambda}_{nm} \bar{x}_{im} - \bar{a}_{in} \in \mathcal{O}(\bar{U}^1_i) Then due to (6) the (\phi_{in})_i$ glue together to give an element $\phi_n \in \mathcal{O}(\bar{A}^1)$. Subtracting for each n an element $\mu_n \in R$ from all the a_{in}'s we may assume $\phi_n(0) = 0$ for all n. Clearly $\phi_1, ..., \phi_g$ are $\bar{K}-$linearly independent. Let us check that they are elements of $F^p\mathbf{X}^1(\bar{A})$; we only have to check that they are additive characters on $\bar{A}^1$. Let V be the kernel of $\bar{A}^1 \to \bar{A}$. We have (8) $\ \phi_n(u+v) = \phi_n(u) + \phi_n(v), \quad u \in \bar{A}^1, v \in V because \bar{\sigma}_i$ is $\mathbf{G}^g_a-$equivariant and transforms the right hand side of (7) into an "affine polynomial" (i.e. an additive polynomial plus a term of degree zero). But property (8) immediately implies that ϕ_n is additive; indeed for any fixed $u \in \bar{A}^1$ the regular function on $\bar{A}^1$ defined by $u \mapsto \phi_n(u+v) - \phi_n(u) - \phi_n(v)$ vanishes at 0 and is constant on the fibres of $\bar{A}^1 \to \bar{A}$. Since $\mathcal{O}(\bar{A}) = \bar{K}$ the above function is 0, hence ϕ_n is additive.

Now, exactly as in Lemma 1, any element $\phi \in F^p\mathbf{X}^1(\bar{A})$ is represented by polynomials H_i of the form $H_i = h_i + \sum_{n=1}^g h_{in0} \bar{x}_n + \sum_{n=1}^g h_{in1} \bar{x}^p_n where h_i, h_{in0}, h_{in1} \in \mathcal{O}(\bar{U}_i)$. As in Lemma 1 one gets that $(h_{in1})_i$ glue together to give an element $h_{n1} \in \bar{K}$. Hence $\phi - \sum_{n=1}^g h_{n1}\phi_n \in F^1\mathbf{X}^1(\bar{A}) so by Lemma 1 \phi - \sum_{n=1}^g h_{n1}\phi_n = 0$ and we conclude that $\phi_1, ..., \phi_g$ generate $F^p\mathbf{X}^1(\bar{A})$ which concludes the proof of Lemma 2.

Let us make the remark (to be used later) that the two Lemmas above do not depend on the fact that our characteristic p situation lifts to characteristic zero. In particular they hold if we replace $\bar{K}$ by a finite separable extension E of it. (We do not need to assume that the derivation on E lifts to a situation in characteristic zero.)

Let us come back to the proof of our Theorem. One of the key steps is the following remark: we know from [B3], Lemma (2.10), p. 73 that the derivation δ of $\mathcal{O}_A$ into the direct image of $\mathcal{O}_{A^1}$ lifts to a derivation $\tilde{\delta}$ of the whole of $\mathcal{O}_{A^1}$, which, via the isomorphisms σ_i corresponds to derivations $\tilde{\delta}_i : \mathcal{O}(U_i)[x_1, ..., x_g] \to \mathcal{O}(U_i)[x_1, ..., x_g] given by formulae of the form \tilde{\delta}_i = \theta_i + \sum_{n=1}^g x_n v_n + \sum_{n=1}^g L_{in}(x_1, ..., x_g) \partial \partial x_n where L_{in}$ are polynomials of degree ≤ 1 in $x_1, ..., x_g$ with coefficients in $\mathcal{O}(U_i)$. This result was proved in [B3] for a ground field rather than for a ground valuation ring but the same arguments go through in our situation. Note that the existence of the lifting $\tilde{\delta}$ is actually a consequence of the Grothendieck-Messing-Mazur theory [MM]; but the additional information that *deg* $L_{in} \leq 1$ provided by [B3] will be crucial below !

By the universality property of canonical prolongations, $\tilde{\delta}$ induces a section $s : A^1 \to A^2$ of the projection $\pi_2 : A^2 \to A^1$ such that the corresponding map $s^* : \mathcal{O}(U_i^2) \to \mathcal{O}(U_i^1)$ maps δx_{in} into $L_{in}(x_{i1}, ..., x_{ig})$. Since the map $s^* : \mathcal{O}(U_{ij}^2) \to \mathcal{O}(U_{ij}^1)$ is the identity on $\mathcal{O}(U_{ij}^1)$ applying this map to (4) we get (9) $\quad \delta a_{ijn} = b_{in} - b_{jn}, \quad b_{in} = L_{in}(x_{i1}, ..., x_{ig})$

In what follows we shall construct an integral basis $\psi_1, ..., \psi_g$ of $\mathbf{X}^2(A)$. Let (λ_{nm}) be the matrix appearing in the proof of Lemma 2 and define (10) $\quad \psi_{in} := \sum_{m=1}^{g} \lambda_{nm}(-\delta x_{im} + b_{im}) \in \mathcal{O}(U_i^2) By (9) (\psi_{in})_i$ glue together to give an element $\psi_n \in \mathcal{O}(A^2)$. Subtracting, for each n, an element $\nu_n \in R$ from all of the b_{in}'s we may assume that $\psi_n(0) = 0$ for all n. Clearly $\psi_1, ..., \psi_g$ are $K-$linearly independent. An argument similar to the one in the proof of Lemma 2 shows that ψ_n are additive characters (instead of $\mathcal{O}(\bar{A}) = \bar{K}$ one uses the fact that $\mathcal{O}(A_K^1) = K$).

To complete the proof of the first part of the Theorem we will check that, with $\phi_1, ..., \phi_g$ as in the proof of Lemma 2 and with $\psi_1, ..., \psi_g$ as in (10) above we have $\bar{\psi}_n = \delta\phi_n$ for all n. Now $\bar{\psi}_{in} - \delta\phi_{in} = \sum \bar{\lambda}_{nm}\bar{b}_{im} + \sum(\delta\bar{\lambda}_{nm})\bar{x}_{im} + \delta\bar{a}_{in} By (9) we have \bar{b}_{im} = \bar{L}_{im}(\bar{x}_{i1}, ..., \bar{x}_{ig}) while on the other hand by (2) we have \delta\bar{a}_{in} = \theta_i\bar{a}_{in} + \sum_{m=1}^{g}(v_m\bar{a}_{in})x_{im} so we see that \bar{\psi}_n - \delta\phi_n \in F^1\mathbf{X}^1(\bar{A}) By Lemma 1 we get \bar{\psi}_n - \delta\phi_n = 0$ which completes the proof of assertion 1).

To prove the second part of the theorem we will relate the Manin map in characteristic p to the p-descent map, by the following construction, which generalizes that of [V1]. The isogeny of $\bar{A}$ to itself defined by "multiplication by p" factors as $V \circ F$ where $F : \bar{A} \to \bar{A}^{(p)}$ is the Frobenius and $V : \bar{A}^{(p)} \to \bar{A}$ is the Verschiebung. Since the latter i s etale, the points of $\ker V$ are rational over the separable closure E of $\bar{K}$. Clearly, $\{x \in E | \delta x = 0\} = E^p$. By Cartier duality $\ker F$ is $E-$isomorphic to μ_p^g, hence (11) $\mathrm{H}^1(E, \ker F) = H^1(E, \mu_p^g) = (E^*/(E^*)^p)^g \hookrightarrow E^g, where \mathrm{H}^1$ stands for the first flat cohomology group of group schemes and the last map is induced by the logarithmic derivative $E^* \to E, x \mapsto \delta x/x$.

Now, the coboundary map in flat cohomology $\bar{A}^{(p)}(E) \to H^1(E, \ker F) = (E^*/(E^*)^p)^g$ can be given by $P \mapsto (f_1(P), \ldots, f_g(P))$, where the functions $f_1, \ldots, f_g$ have divisors $pD_1, \ldots, pD_g$, such that $D_1, \ldots, D_g$ form a basis for the p-torsion of the Picard variety of $\bar{A}^{(p)}$. Indeed, given such a divisor D_i, we get a map from $\mathbf{Z}/p\mathbf{Z}$ to the p-torsion of the Picard variety of $\bar{A}^{(p)}$ and, by duality, a map from $\ker F$ to μ_p. Also $f_i \circ [p] = g_i^p$ for some function g_i and the map $H^1(E, \ker F) \to (E^*/(E^*)^p)^g$ is given by associating the torsor $F^{-1}(P)$ of $\ker F$ to the torsor $g_i(F^{-1}(P))$ of μ_p and, clearly, this torsor corresponds to $f_i(P)$.

Therefore the map $\beta : \bar{A}^{(p)}(E) \to E^g$, obtained by composing the coboundary map with the logarithmic derivative (see (11)), is given by δ-polynomial characters of order 1 and degree 1. Also, by construction the kernel of β is $F(\bar{A}(E))$. Now, the proof of the proposition in section 4 of [V2] shows that the matrix formed by the images of an $\mathbf{F}_p$-basis of $\ker V$ in $(E^*/(E^*)^p)^g$ is essentially the matrix of the Serre-Tate parameters of $\bar{A}$ (modulo p-th powers) at any place of $\bar{K}$ of good, ordinary, reduction for $\bar{A}$. Also, a theorem of Katz ([K2]) shows that its image in E^g is the matrix of the cup-product with the Kodaira-Spencer class and therefore, is of maximal rank. (This argument is given in detail in [V2]). Let $\wp : E^g \to E^g$ be an additive polynomial map whose kernel is the image of $\ker V$; then $\wp$ is clearly of degree p. We now define $\mu : \bar{A}(E) \to E^g$ by $\mu(P) = \wp(\beta(Q))$, where $Q \in \bar{A}^{(p)}(E)$ is such that $V(Q) = P$. It is clear that the definition is independent of the choice of Q and is given by δ-polynomial characters of order 1 and degree p on $\bar{A}$. By Lemma 2 (and the remark after it) the components of μ are defined by $E-$linear combinations of $\phi_1, ..., \phi_g$. Let us show that the kernel of μ is $p\bar{A}(E)$. Suppose $\mu(P) = 0$, then after changing Q by an element of $\ker V$, we get $\beta(Q) = 0$, so $Q = F(R), R \in \bar{A}(E)$, hence $P = V(F(R)) = pR$.

Now the condition $\ker \mu = p\bar{A}(E)$ implies that $\{P \in \bar{A}(E) | \hat{\phi}_1(P) = \ldots = \hat{\phi}_g(P) = 0\} = p\bar{A}(E). Indeed, since the \phi_i$'s are a basis of $F^p\mathbf{X}^1(\bar{A})$, the left hand side contains $\ker \mu = p\bar{A}(E)$ and the reverse inclusion is clear since E has characteristic p. We get that $\{P \in \bar{A}(\bar{K}) | \hat{\phi}_1(P) = \ldots = \hat{\phi}_g(P) = 0\} = p\bar{A}(E) \cap \bar{A}(\bar{K}) We claim that p\bar{A}(E) \cap \bar{A}(\bar{K}) = p\bar{A}(\bar{K}) which will close the proof. To check the claim note that the proposition in [V2] says that \bar{A}(E)$ has no point of order p; so if $P = pQ \in \bar{A}(\bar{K})$ for some $Q \in \bar{A}(E)$, then for all $\sigma \in \mathrm{Gal}(E/\bar{K})$ we have $p(Q - Q^\sigma) = 0$ hence $Q - Q^\sigma = 0$ so $Q \in \bar{A}(\bar{K})$ and we are done.

Remark. Assume we are in the hypotheses of the Theorem. Let $\nabla_\delta : H^1_{DR}(A) \to H^1_{DR}(A) betheadditivemapobtainedby$ $Maninconnectionat\delta$ and view $H^0(A, \Omega^1)$ as embedded into $H^1_{DR}(A)$. Let $\omega = (\omega_1, ..., \omega_g)$ be an $R-$basis of $H^0(A, \Omega^1)$ and write the Picard Fuchs equation: (12) $\quad \nabla^2_\delta \omega + \alpha \nabla_\delta \omega + \beta\omega = 0 where \alpha, \beta$ are $g \times g$ matrices with entries in R. Moreover let $\bar{e}_1, ..., \bar{e}_g$ be the image of $\bar{\omega}_1, ..., \bar{\omega}_g \in H^0(\bar{A}, \Omega^1)$ via the isomorphism $\rho(\delta) : H^0(\bar{A}, \Omega^1) \to H^1(\bar{A}, \mathcal{O})$ given by the cup product with the Kodaira Spencer class $\rho(\delta) \in H^1(\bar{A}, T)$ and consider the Hasse-Witt matrix $\bar{\lambda}$ with respect to the basis $\bar{e}$, in other words write $\bar{e}^{(p)} = \bar{\lambda}\bar{e}$. Then we claim that $\bar{\lambda}$ satisfies the following "dual Picard-Fuchs equation": (13) $\quad \delta^2\bar{\lambda} - (\delta\bar{\lambda})\bar{\alpha} + \bar{\lambda}(\bar{\beta} - \delta\bar{\alpha}) =$ $0 The above claim is a dual version of the Igusa-Manin-Katz theorem referred to in the beginning of the paper.$

To check the claim above, let $\psi_1, ..., \psi_g$ be as in the Theorem and let us borrow the notations from the proof of the Theorem. The ψ_i's define an $R-$group scheme homomorphism $\psi : A^2 \to \mathbf{G}^g_a$. We have induced maps $\mathrm{H}^0(A, \Omega^1) = L(A)^\circ \subset L(A^2)^\circ \psi^* \leftarrow L(\mathbf{G}^g_a)^\circ = R^g where the upper \circ$ means "dual". Recall from [B2] Part III that one has natural induced derivations $\mathrm{L(A)}^\circ \delta \to L(A^1)^\circ \delta \to L(A^2)^\circ \delta \to ... and that there exists a basis \mathrm{z} = (\mathrm{z}_1, ..., z_g)$ of R^g such that $\psi^* z = \delta^2\omega + \alpha\delta\omega + \beta\omega where \alpha, \beta$ are the matrices appearing in (11). (Recall that this was deduced in [B2], Part III, (3.6) as a consequence of the Grothendieck-Messing-Mazur theory [MM], cf. also [B3], Chapter 2). Moreover $\omega, \delta\omega$ form a basis of $L(A^1)^\circ$ while $\omega, \delta\omega, \delta^2\omega$ form a basis of $L(A^2)^\circ$.

Now the Theorem implies that the map $\bar{\psi}^* : \bar{K}^g \to L(\bar{A}^2)^\circ$ agrees with the composition $\bar{K}^g \phi^* \to L(\bar{A}^1)^\circ \delta \to L(\bar{A}^2)^\circ onth$ $((1, 0, ..., 0), (0, 1, ..., 0), ...)$ of $\bar{K}^g$. Write $\bar{z} = \bar{\xi}\epsilon$ where ξ is an invertible matrix with entries in R and write $\phi^*\epsilon = \bar{\gamma}\delta\bar{\omega} + \bar{\eta}\bar{\omega}$ where γ, η are $g \times g$ matrices with entries in R. By (7) we have $\bar{\gamma} = -\bar{\lambda}$. We get arraylll $\delta^2\bar{\omega} + \bar{\alpha}\delta\bar{\omega} + \bar{\beta}\bar{\omega} = \psi^*(\bar{\xi}\epsilon) = \bar{\xi}\psi^*\epsilon = \bar{\xi}\delta\phi^*\epsilon$

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