Differential operators and interpolation series in power series fields

José Felipe Voloch

Let k be a field and K be the field of formal power series over k. That is, the elements of K are of the form $u = \sum_{n=n_0}^{\infty} a_n x^n$, where $a_n \in k$ and n_0 is an arbitrary integer. If $a_{n_0} \neq 0$ we put $v(u) = n_0$, then v is a valuation on K and K is a local field, i.e., it is complete with respect to this valuation. Let U be an open subset of K and $f : U \to K$ a function. Besides the usual notion of continuity there is the notion of differentiability for such functions f, namely, f is differentiable in $a \in U$ if $\lim_{u \to a}(f(u) - f(a))/(u - a)$ exists. A natural class of functions to consider is that of differential operators, coming from differentiation with respect to the variable x. We can define the Hasse derivations $D^{(r)}, r \geq 0$ by:

$$D^{(r)}(\sum a_n x^n) = \sum \binom{n}{r} a_n x^{n-r}.$$

Theorem 1. *The functions $D^{(r)} : K \to K, r \geq 1$ are k-linear, continuous and nowhere differentiable. (Differentiation is not differentiable!)*

Proof: It is clear that $D^{(r)}$ is k-linear and therefore it suffices to check continuity and differentiability at $u = 0$. Plainly $v(D^{(r)}(u)) \geq v(u) - r$, so $D^{(r)}$ is continuous (see also [Go], Prop. 13). Next, $D^{(r)}$ is differentiable at $u = 0$ if and only if $\lim_{u \to 0} D^{(r)}(u)/u$ exists. However, the sequence x^n converges to 0 but $D^{(r)}(x^n)/x^n = \binom{n}{r} x^{-r}$ does not converge.

Suppose now that k is a finite field with q elements. Then Wagner [W] studied continuous linear functions $f : R \to K$, where $R = k[[x]]$. He obtained results analogous to classical results of Mahler [M] that gave interpolation series for continuous p-adic functions in terms of binomial coefficients. To state Wagner's result we need to make a few definitions:

$$\Psi_n(u) = \prod_{\substack{m \in k[x] \\ \deg m < n}} (u - m), n > 0, \Psi_0(u) = u,$$

$$F_n = (x^{q^n} - x)(x^{q^{n-1}} - x)^q \cdots (x^q - x)^{q^{n-1}}, F_0 = 1,$$

$$L_n = (x^{q^n} - x)(x^{q^{n-1}} - x) \cdots (x^q - x), L_0 = 1.$$

Wagner then proved that every continuous linear function $f : R \to K$ can be written as $f = \sum_{n=0}^{\infty} A_n \Psi_n / F_n$,where $A_n \in K, \lim_{n\to\infty} A_n = 0$ and moreover f is differentiable if and only if $\lim_{n\to\infty} A_n / L_n = 0$. The A_n can be obtained as follows. Define:

$$\Delta_0 f(u) = f(u)$$

$$\Delta_{n+1} f(u) = \Delta_n f(xu) - x^{q^n} \Delta_n f(u).$$

Wagner then shows that $A_n = \Delta_n f(1)$. Our next result computes the A_n for the $D^{(r)}$.

Theorem 2. *For all $u \in R$ we have:*

$$D^{(r)}(u) = \sum_{n=0}^{\infty} A_{nr} \frac{\Psi_n(u)}{F_n},$$

where $A_{n1} = (-1)^{n-1} L_{n-1}$ and, for $r > 1$,

$$A_{nr} = (-1)^{n-1} L_{n-1} \sum_{0<i_1<\cdots<i_{r-1}<n} \frac{1}{(x - x^{q^{i_1}}) \cdots (x - x^{q^{i_{r-1}}})}.$$

Proof: We will show that $\Delta_n D^{(r)} = \sum_{i=0}^{r-1} A_{n,r-i} D^{(i)}$, for $n \geq 1$, by induction on n, and the result will follow from Wagner's results. Clearly, $\Delta_1 D^{(r)} = D^{(r-1)}$ so the above formula holds for $n = 1$. Assume the formula holds for n. From the recursive definition of Δ_{n+1} we get that

$$\Delta_{n+1} D^{(r)} = \sum_{i=0}^{r-1} (A_{n,r-i}(x - x^{q^n}) + A_{n,r-i-1}) D^{(i)} = \sum_{i=0}^{r-1} A_{n+1,r-i} D^{(i)}$$

and this completes the proof.

In particular we get the bizarre formula $du/dx = \sum_{n=0}^{d} (-1)^{n-1} L_{n-1} \Psi_n(u) / F_n$ for $u \in k[x], \deg u \leq d$.

Another class of continuous linear functions are $u \mapsto u \circ b$ for $b \in xR$. These are differentiable when $b = x$ or $v(b) > 1$. The coefficients of their expansion in Wagner's basis are given by $(b - x) \cdots (b - x^{q^{n-1}})$. The proof is left to the reader.

Finally we establish the following formula expanding the functions u^{q^i} in terms of the Hasse derivatives.

Proposition. *We have*

$$u^{q^i} = \sum_{r=0}^{\infty} (x^{q^i} - x)^r D^{(r)} u$$

for $u \in k[[x]]$.

Proof: We begin with the case $i = 1$; i.e.,

$$u^q = \sum_{r=0}^{\infty} (x^q - x)^r D^{(r)} u \, .$$

Note that both sides of this equation are k-linear, so it suffices to check the formula for $u = x^m$ and in this case it is straightforward. In this formula for u^q, one can replace q by q^n, for any n by extending k to its extension of degree n. The proposition now follows.

Acknowledgments: The author would like to thank D. Goss for suggestions and encouragement. The author would also like to acknowledge financial support by the NSF (grant no. DMS-9301157) and by the NSA (grant MDA904-97-1-0037).

References

[Go] D. Goss, *Separability, multi-valued operators, and zeroes of L-functions*, preprint, 1997.

[M] K. Mahler, *Introduction to p-adic numbers and their functions*, Cambridge Tracts in Mathematics **64**, Cambridge Univ. Press, Cambridge, 1973.

[W] C. G. Wagner, *Linear operators in local fields of prime characteristic* , Crelle **251**, 153-160.

Dept. of Mathematics, Univ. of Texas, Austin, TX 78712, U.S.A.