Distance functions on varieties over non-archimedean local fields

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Let K be a field, complete with respect to an absolute value $|.|$. Let X be an algebraic variety defined over K. Then $X(K)$ has a natural topology coming from the topology of K induced by $|.|$ and it is possible, in many different ways, to describe this topology by a metric. This has been studied, for instance, in [Si], where many functorial properties are obtained. However, the main focus of [Si] is to obtain global height functions, so the metrics considered are defined in completions of global fields and care was used to study how things varied with respect to the place. Also, archimedean valuations are considered. All this results in the results of [Si] stated as holding only modulo bounded functions. Some of the results of [Si] were extended in [B]. Another approach, for projective space, was suggested in [R] and studied further in [CV]. Again, the focus was on global fields. The purpose of this note is to concentrate on the non-archimedean local case and obtain more refined results. We will use a different development of the theory, but we will show that we recover the metrics defined by the aforementioned authors, in particular showing that they coincide, which is not obvious from their definitions. Our results will be sharp and will not involve any unspecified bounded function. Most, if not all, of our results will not be a surprise to the experts and, in fact, we have implicitly used these results, e.g. in [V]. Nevertheless, it seems appropriate to record these results with proofs, since no other source is currently available in the literature. At the end we will discuss some global results and give a sharpening of a result of Carlitz which suggests an interesting conjecture.

In the case $K = \mathbf{Q}_p$, the definition of our metric will be, informally, that $d(P, Q) = p^{-m}$, if P, Q are equal modulo p^m but not modulo p^{m+1}. To make sure that our definitions are independent of any choice of coordinates and to deal with more general situations, we will use the language of schemes. Perhaps with a bit more effort one could dispense with that.

Let K be a field, complete with respect to an absolute value $|.|$. That is, for any

$x \in K, |x|$ is a real number and the following holds:

(i) For all $x \in K, |x| \geq 0$ and $|x| = 0$ if and only if $x = 0$.

(ii) For all $x, y \in K, |xy| = |x||y|$.

(iii) For all $x, y \in K, |x + y| \leq \max\{|x|, |y|\}$, with equality holding if $|x|, |y|$ are distinct.

Let $\mathcal{O} = \{x \in K \mid |x| \leq 1\}$, be the ring of integers of K and, for any $\epsilon, 0 < \epsilon \leq 1$, define $\mathcal{M}_\epsilon = \{x \in K \mid |x| < \epsilon\}$. So, $\mathcal{M}_1 = \mathcal{M}$ is the maximal ideal of $\mathcal{O}$. Let $\mathcal{O}_\epsilon = \mathcal{O}/\mathcal{M}_\epsilon$ for $0 < \epsilon \leq 1$. If X is a scheme over $\mathcal{O}$, we will denote by X_ϵ the base change of X to $\mathcal{O}_\epsilon, X_\epsilon = X \otimes_{\mathcal{O}} \mathcal{O}_\epsilon$.

Let X be a reduced scheme of finite type over $\operatorname{Spec} \mathcal{O}$. We will define a metric on $X(\mathcal{O})$. More generally, we will define, for a closed subscheme Y of X a distance function of elements of $X(\mathcal{O})$ to Y as follows. Let $d(P, Y) = \inf\{\epsilon > 0 \mid P_\epsilon \in Y_\epsilon\}$, provided this set is non-empty, otherwise set $d(P, Y) = 1$.

If we start with a variety over K we can always take a model over $\mathcal{O}$ and work there. If the variety is projective, then any K-rational point extends to an integral point of the integral model. In general, this will not be the case, and thus our metric is only defined for integral points. This is similar to a difficulty encountered in [Si] where in the quasi-projective case, he obtained his distance functions modulo a "distance to the boundary".

Theorem 1. *Let X be a reduced scheme of finite type over* $\operatorname{Spec} \mathcal{O}$*. The distance defined above has the following properties:*

(a) If Y is an effective Cartier divisor on X and U is an open subset of X on which Y is given by $(f), f \in \mathcal{O}_X(U)$, then $d(P, Y) = |f(P)|$ for $P \in U(\mathcal{O})$.

(b) If Y, Z are closed subschemes of X then $d(P, Y \cap Z) = \max\{d(P, Y), d(P, Z)\}$.

(c) If Y is another reduced scheme of finite type over $\operatorname{Spec} \mathcal{O}$ *and $f : X \to Y$ is a morphism of* $\operatorname{Spec} \mathcal{O}$*-schemes, then for any closed subscheme Z of Y, $d(P, f^*(Z)) = d(f(P), Z)$, for all $P \in X(\mathcal{O})$.*

(d) As a function on $X(\mathcal{O}) \times X(\mathcal{O}), d(P, Q)$ defines a metric which induces the topology coming from the topology of $\mathcal{O}$. Moreover, it satisfies the ultrametric inequality $d(P, R) \leq \max\{d(P, Q), d(Q, R)\}$, for all $P, Q, R \in X(\mathcal{O})$ with equality holding if

$d(P,Q), d(Q,R)$ are distinct.

(e) *If L/K is a Galois extension with ring of integers $\mathcal{O}'$, $\sigma \in \mathrm{Gal}(L/K)$ and Y is a closed subscheme of $X \otimes_{\mathcal{O}} \mathcal{O}'$, we have $d(P^\sigma, Y^\sigma) = d(P, Y)$, for all $P \in X(\mathcal{O}')$.*

Proof: To prove (a), note that the reduction of f modulo $\mathcal{M}_\epsilon$ defines Y_ϵ in U_ϵ and that $|f(P)| < \epsilon$ if and only if the reduction of $f(P)$ modulo $\mathcal{M}_\epsilon$ is zero. Likewise, (b) is straightforward. For (c), it suffices to notice that $f(P_\epsilon) = f(P)_\epsilon$ and that $f^*(Z)_\epsilon = f^*(Z_\epsilon)$, as follows from the functorial properties of base change. For $P, Q \in X(\mathcal{O}), P_\epsilon \in Q_\epsilon$ if and only if $P_\epsilon = Q_\epsilon$, so the ultrametric inequality follows from transitivity of equality and the case of equality in the ultrametric inequality is a formal consequence of it. Finally, it easy to show that the induced topology is that coming from the topology of $\mathcal{O}$, by taking local coordinates and using (a). To prove (e) it is enough to notice that the ideals $\mathcal{M}'_\epsilon$ of $\mathcal{O}'$ are Galois invariant since K is complete and the rest follows from the functorial properties of base change.

Remark: It follows from items (a) and (b) of the theorem and [Si], theorems 1.1 and 2.1 that our distance coincides with that of [Si]. Note however that [Si] always works with $-\log d$.

The distance of [R] is defined as follows. If P, Q are points in $\mathbf{P}^n(K)$ represented by vectors $x, y \in K^{n+1}$, then $d'(P,Q) = |x \wedge y|/|x||y|$, where vectors are given the sup-norm. When regarding a point in $\mathbf{P}^n(K)$ as a point in $\mathbf{P}^n(\mathcal{O})$, one chooses a representative $x \in \mathcal{O}^{n+1}$ one of whose coordinates is a unit, thus $|x| = 1$. If such representatives x, y are chosen for P and Q, then $d'(P,Q) = \max_{i \neq j}\{|x_i y_j - x_j y_i|\}$. To show that this coincides with our definition, note that (compare [Si], §3) $d(P,Q) = d((P,Q), \Delta)$, where Δ is the diagonal in $\mathbf{P}^n \times \mathbf{P}^n$. Now, it follows that $d = d'$ since $x_i y_j - x_j y_i = 0, i \neq j$, gives a system of equations defining Δ. Theorem 1 (d) also gives that, if A is a linear map on $\mathbf{P}^n$ over $\mathcal{O}$, then A induces an isometry on $\mathbf{P}^n(\mathcal{O})$, which is a special case of Theorem 3 of [CV], since A is defined over $\mathcal{O}$ if and only if $\eta(A) = 1$ in the notation of [CV]. More generally, automorphims of $X/\operatorname{Spec}\mathcal{O}$ are isometries.

Another equivalent way of defining the distance is through intersection theory. This is standard when the valuation on K is discrete, but it does work in general. Namely, if $P \in X(\mathcal{O})$ is viewed as a section $s_P : \operatorname{Spec}\mathcal{O} \to X$ and Y is as above, then either the image of s_P and Y are disjoint (in which case $d(P, Y) = 1$) or the image of s_P is contained in Y (in which case $d(P, Y) = 0$) or $s_P^*(Y)$ is a closed subscheme of $\operatorname{Spec}\mathcal{O}$ supported at the closed point, thus defined by an ideal I of $\mathcal{O}$, in which case $d(P, Y) = \inf\{\epsilon \mid I \subset \mathcal{M}_\epsilon\}$, as is readily checked. This shows that $d(P, Y) = 0$ if and only if $P \in Y(\mathcal{O})$.

As an application we prove the following higher dimensional generalization of Krasner's lemma:

Corollary 1. *Let X be a reduced scheme of finite type over* $\operatorname{Spec}\mathcal{O}$*. If P and Q are integral points of X defined over a separable algebraic extension of K and $d(P, Q) < d(P, P')$ for every conjugate P' of P over K, then P is defined over $K(Q)$.*

Proof: If P is not defined over $K(Q)$, then there exists σ in the absolute Galois group of $K(Q)$ such that $P^\sigma \neq P$. It follows from Theorem 1 that

$$d(P, P^\sigma) \leq \max\{d(P, Q), d(Q, P^\sigma)\} = d(P, Q)$$

and this contadicts the hypothesis.

Corollary 2. *Let G be a reduced group-scheme of finite type over* $\operatorname{Spec}\mathcal{O}$*. Then the distance is translation invariant.*

Proof: This follows immediately from Theorem 1 (c), applied to the morphism given by translation by an element of $G(\mathcal{O})$.

Remark: It follows that, in the situation of the corollary, $d(P, Q) = d(PQ^{-1}, 1)$ and, from the ultrametric inequality, the sets $G^{(\epsilon)} = \{P \in G(\mathcal{O}) \mid d(P, 1) < \epsilon\}$ form a filtration of $G(\mathcal{O})$ by subgroups, which recovers the canonical filtration of the formal group $G^{(1)}$ of G.

We should also remark that, given a distance function on points, there is an obvious alternative way of defining distances to subschemes as follows: $d'(P, Y) = \inf_{Q \in Y(\mathcal{O})} d(P, Q)$. It follows from theorem 1 that $d(P, Y) \leq d(P, Q)$ for $Q \in Y(\mathcal{O})$ by applying item (b) with

$Z = Q$, so $d(P, Y) \leq d'(P, Y)$ for all $P \in X(\mathcal{O})$. If the valuation on K is discrete then, by [G1], corollary 1, there exists $c, \delta > 0$, such that $d'(P, Y) \leq cd(P, Y)^\delta$, for all $P \in X(\mathcal{O})$ and this was generalized to arbitrary K as above in [Sc].

As a final application we discuss a global problem in "non-linear diophantine approximation".

Let X be an algebraic variety defined over a field K, either a number field or a function field in one variable, with field of constants k. Let v be a place of K and Y a subvariety of X defined over the local field K_v. We shall be interested in points in $X(K)$ which are v-adically close to Y in terms of their heights. If Y is defined over K, it is easy to get an estimate that states that a point cannot be too close to Y unless it is in Y, which can be viewed as a generalization of Liouville's theorem in diophantine approximation. When K is a number field, one can view Vojta's conjectures [Vo] as an analogue of Roth's theorem in this context, but these remain largely unproved, except when X is an abelian variety. Our goal is to state and prove, under some hypotheses in the function field case, a result which can be viewed as an analogue of Dirichlet's theorem. Such results are known if X is projective space and Y is a linear subspace, whereas we try to obtain results in a more general context, which explains the term "non-linear". Such questions seem to have first been raised by Schmidt [S1] and he obtained some results for hypersurfaces of projective space of sufficiently high dimension in the number field case in [S2]. The special case of projective quadrics has been extensively studied under the guise of Oppenheim's conjecture and there the results are sharper in the number field case, due to the work of Margulis and others (see [M]).

Let $K = k(t)$, where k is an algebraically closed field and t is an indeterminate and consider its completion $K_v = k((t))$ with respect to the valuation v centered at 0. We consider the distance function, as defined above, for varieties over K_v. We define the global height of $P = (x_0 : \ldots : x_n) \in \mathbf{P}^n(K)$ as $h(P) = \max\{\deg x_i\}$ if $x_0, \ldots, x_n \in k[t]$ have no common factor.

Theorem 2. *Let* $f_1, \ldots, f_s \in K[x_0, \ldots, x_n], g \in K_v[x_0, \ldots, x_n]$ *be homogeneous polyno-*

mials. Let $X = \{P \in \mathbf{P}^n \mid f_1(P) = \cdots = f_s(P) = 0\}, Y = \{P \in X \mid g(P) = 0\}$*. Let* $d_i = \deg f_i, e = \deg g$*. Assume that* $\sum d_i + e \leq n$ *and that* $Y(K) = \emptyset$*. Then, for infinitely many* $P \in X(K), -\log(d(P,Y)) \geq \mu h(P) + O(1)$*, where* $\mu = n + 1 - \sum d_i$*.*

Proof: The proof is an extension of the classical proof of Tsen's theorem (compare, e.g. [G2]). A similar argument occurs in [C], where finite, instead of algebraically closed fields are considered. However, in [C], it is only proved that there is a solution to the given inequality, as opposed to infinitely many.

Let H be a large integer and consider $x_0, \ldots, x_n \in k[t]$ of degree at most H, polynomials to be determined. Their coefficients then give $(n+1)(H+1)$ unknowns in k. The conditions $f_i(x_0, \ldots, x_n) = 0, i = 1, \ldots, s$ and $v(g(x_0, \ldots, x_n)) \geq M$ impose $H \sum_1^s d_i + M + O(1)$ conditions given by homogeneous equations in the unkowns. A solution can be guaranteed to exist if we let $M = H(n + 1 - \sum_1^s d_i) + O(1)$. Let $x_0, \ldots, x_n$ be a solution and d their greatest common divisor. Then putting $P = (x_0 : \ldots : x_n)$ we have $h(P) = \max\{\deg x_i\} - \deg d \leq H - \deg d$ and $-\log d(P,Y) \geq M - v(d)e \geq M - \deg de \geq \mu h(P) + O(1)$. What remains to be shown is that this process leads to infinitely many distinct points P. Suppose not. Assume that the same point P gives rise to infinitely many $x_0, \ldots, x_n$ and therefore to infinitely many d. From $v(g(x_0, \ldots, x_n)) \geq M$ we get that, since $g(x_0, \ldots, x_n) \neq 0$, $ev(d) \geq M + O(1)$ hence $e \deg d \geq M + O(1)$, and since $\deg d \leq H$ we get $eH \geq M + O(1)$, dividing by H and letting H go to infinity then gives $e \geq n + 1 - \sum_1^s d_i$, contradicting the hypothesis. This completes the proof.

This result is best possible in its generality. For example, let $n = 1, s = 0$ and $g(x_0, x_1) = x_0^2 + tx_1^2$, then $X = \mathbf{P}^1$ and $-\log d(P,Y) \leq 1, \forall P \in X$. There are other examples in all dimensions. It is conceivable that under additional hypotheses there will be a similar result for higher degree. The same polynomial g can be used for $n = 2$, say, to show that the constant μ cannot be improved in general. However, one may conjecture that if $Y(K_v)$ is Zariski dense in Y, one may weaken the restriction $\sum d_i + e \leq n$ to $\sum d_i \leq n$. One may also conjecture that similar statements also hold in the number field case.

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