

## Periods of abelian varieties in characteristic $p$

**José Felipe Voloch**

If  $A$  is an abelian variety defined over the complex numbers  $\mathbf{C}$ , then there exists a lattice  $\Lambda \subset \mathbf{C}^g, g = \dim A$ , such that  $A(\mathbf{C}) = \mathbf{C}^g/\Lambda$ . This lattice is called the period lattice because functions on  $A$  will be periodic functions on  $\mathbf{C}^g$  with periods in  $\Lambda$ . In this note we give an analogue in characteristic  $p$  for the period lattice  $\Lambda$  and for the parametrization  $A(\mathbf{C}) = \mathbf{C}^g/\Lambda$ .

Notation: For an abelian group  $H$  we define  $\hat{H} = \varprojlim H/p^n H$ . Then  $\hat{H}$  is a  $\mathbf{Z}_p$ -module.

Let  $A$  be an ordinary abelian variety over a field  $K$  of characteristic  $p > 0$  and let  $K_s$  be the separable closure of  $K$  and  $G = \text{Gal}(K_s/K)$ . Let  $A^{(p^n)}$  be the image of  $A$  under the  $n$ -th power of the Frobenius map  $F^n$  and  $V_n : A^{(p^n)} \rightarrow A$  the dual isogeny, the  $n$ -th order Verschiebung, which is separable since  $A$  is ordinary. Then  $\ker V_n$  is the  $p^n$  torsion of  $A^{(p^n)}$ . We define the period lattice by  $\Lambda = \varprojlim \ker V_n$ . This definition is not new, it corresponds to the Serre-Tate parameters (see e.g. [K]), however it usually only considered when the ground field is a local field. See [K] also for the relationship between the Serre-Tate parameters and moduli. The generalization of the analytic parametrization of an abelian variety is given by the following:

**Theorem 1.**  $\Lambda \rightarrow \widehat{K_s^*} \otimes \Lambda^{\otimes(-1)} \rightarrow \widehat{A(K_s)} \rightarrow 0$  as  $G$ -modules.

A few comments are in order. As a  $\mathbf{Z}_p$ -module,  $\widehat{K_s^*} \otimes \Lambda^{\otimes(-1)}$  is isomorphic to  $(\widehat{K_s^*})^g, g = \dim A$ , but they are different as  $G$ -modules. Secondly, the analogy with the analytic parametrization is more evident after composing it with the exponential map. This construction also generalises the Tate parametrization of elliptic curves: If  $K_v$  is a local field and  $E/K_v$  has split multiplicative reduction then  $\exists q \in K_v$  such that  $0 \rightarrow q^{\mathbf{Z}} \rightarrow K_v^* \rightarrow E(K_v) \rightarrow 0$  ([S], Ch. V). It is easy to show that Theorem 1 follows from the Tate parametrization in this case (see [V1], lemma 2). This generalizes to Mumford's parametrization of abelian varieties with completely multiplicative reduction [Mu].

*Proof:* We we have the exact sequence of group schemes

$$0 \rightarrow \ker F^n \rightarrow \ker[p^n] \rightarrow \ker V_n \rightarrow 0.$$

Taking flat cohomology yields

$$\ker V_n(K_s) \rightarrow H^1(K_s, \ker F^n) \rightarrow H^1(K_s, \ker[p^n]) \rightarrow 0.$$

On the other hand,  $H^1(K_s, \ker[p^n]) = A(K_s)/p^n A(K_s)$ , which follows from the exact sequence  $0 \rightarrow \ker[p^n] \rightarrow A \rightarrow A \rightarrow 0$  and also

$$H^1(K_s, \ker F^n) = H^1(K_s, \mu_{p^n}) \otimes \ker V_n^{\otimes -1} = K_s^*/(K_s^*)^{p^n} \otimes \ker V_n^{\otimes -1}.$$

Putting these together and passing to the inverse limit yields the theorem.

**Corollary.** *If  $K$  is a global field,  $E/K$  an elliptic curve and  $v$  a place of  $K$  where  $E$  has bad reduction, then  $q$  is transcendental over  $K$  and so is any  $u \in K_v^*$  which maps to a point of infinite order in  $E(K)$ .*

This corollary is proved in detail in [V]. The proof consists in comparing the Tate parametrization and the parametrization given by Theorem 1 and using a theorem of Igusa which guarantees that the action of  $G$  on  $\Lambda$  is not through a finite quotient. The transcendence of  $q$  is the characteristic  $p$  analogue of the recent result of Barré-Sirieix et al. [B]. It would be nice to generalize the corollary to higher dimensional abelian varieties. This would require understanding the action of  $G$  on  $\Lambda$ . The result follows, for example if  $G$  acts via the full general linear group, which is the generic case by [FC], Prop. V.7.1.

Another application of theorem 1 is to local duality. It is a classical result of Tate (up to the  $p$ -part in characteristic  $p$ , which is due to Milne) that, if  $K$  is a local field with finite residue field, then  $A(K)$  and  $H^1(G, A(K_s))$  are Pontrjagin duals. There is a conjecture of Milne ([M], III, Conjecture 10.7) which generalizes the local duality to case of algebraically closed residue field. This conjecture is known in the case of good reduction (Bester, see [M]) and for elliptic curves with split multiplicative reduction (Shatz, see [M]). We extend these results a bit in the following proposition, and we believe its proof may be extended to give further results along these lines.

**Proposition.** *Let  $K$  be a local field whose residue field is the algebraic closure of a finite field of characteristic  $p > 0$  and  $A/K$  an abelian variety whose reduction is a semi-abelian variety with ordinary abelian quotient. Assume also that  $A[p] \cap A(K_s) = 0$ , then  $T_p(H^1(G, A(K_s)))$  is isomorphic to  $H^1(G, \Lambda)$ .*

*Proof:* Consider the exact sequence of  $G$ -modules

$$0 \rightarrow K_s^* \xrightarrow{p^n} K_s^* \rightarrow K_s^*/(K_s^*)^{p^n} \rightarrow 0.$$

Under the hypotheses of the theorem,  $H^1(G, K_s^*) = H^2(G, K_s^*) = 0$ , so the Galois cohomology sequence of the above exact sequence yields  $(\widehat{K_s^*})^G = \widehat{K^*}$  and  $H^1(G, \widehat{K_s^*}) = 0$ .

Now consider the exact sequence of  $G$ -modules

$$0 \rightarrow A(K_s) \xrightarrow{p^n} A(K_s) \rightarrow A(K_s)/p^n A(K_s) \rightarrow 0.$$

Taking Galois cohomology yields  $0 \rightarrow \widehat{A(K)} \rightarrow \widehat{A(K_s)}^G \rightarrow T_p(H^1(G, A)) \rightarrow 0$ .

By our hypothesis on the reduction type of  $A$  we obtain that  $V_n$  is étale on the special fibre as well as on  $A$ , thus  $\Lambda = \mathbf{Z}_p^g$  with the trivial action of  $G$ . It also follows that  $V_n$  is an isomorphism on the formal group of  $A$ . Thus, given  $P \in A(K)$  in the formal group, we can find  $Q \in A^{(p^n)}(K)$ ,  $V_n(Q) = P$  and we can then map  $Q$  to  $H^1(K, \ker F^n)$  using the coboundary map of the flat cohomology sequence coming from the exact sequence  $0 \rightarrow \ker F^n \rightarrow A \rightarrow A^{(p^n)} \rightarrow 0$ . This gives us an inverse, in the formal group, to the map  $(\widehat{K^*})^g = \varprojlim H^1(K, \ker F^n) \rightarrow \widehat{A(K)}$  which comes from theorem 1. It follows that  $\widehat{A(K)} = (\widehat{K^*})^g/\Lambda$ .

We are now ready to take Galois cohomology of the exact sequence of theorem 1. Note that under our present assumptions this sequence is exact on the left also. We get

$$0 \rightarrow \Lambda \rightarrow (\widehat{K^*})^g \rightarrow \widehat{A(K_s)}^G \rightarrow H^1(G, \Lambda) \rightarrow 0.$$

Since  $(\widehat{K^*})^g$  surjects onto  $\widehat{A(K)}$ , we obtain that

$$T_p(H^1(G, A)) = \widehat{A(K_s)}^G / \widehat{A(K)} = H^1(G, \Lambda).$$

We say that an ordinary abelian variety  $A$  is sufficiently general if  $A[p^\infty] \cap A(K_s)$  is finite. It follows from the proof of theorem 1, that  $A$  is sufficiently general if and only if the map  $\Lambda \rightarrow \widehat{K_s^*} \otimes \Lambda^{\otimes(-1)}$  is injective. In [V2] a sufficient condition for  $A$  to be sufficiently general is given which justifies the name "sufficiently general". The following theorem studies the action of the endomorphisms of  $A$  on  $\Lambda$  and produces a best possible result under the hypotheses, showing that  $\Lambda$  behaves like the period lattice in this case and also like the  $\ell$ -adic representation.

**Theorem 2.** *The natural map  $\text{End}(A) \otimes \mathbf{Z}_p \rightarrow \text{End}(\Lambda)$  is injective if  $A$  is sufficiently general.*

*Proof:* It suffices to show, by standard arguments, that if  $\phi \in \text{End}(A)$  acts trivially (via  $\phi^{(p^n)}$ ) on  $\ker V_n$  for  $n$  large, then  $\phi$  factors through  $[p] : A \rightarrow A$ . Let  $\check{A}$  be the dual abelian variety and fix a polarization  $\alpha : A \rightarrow \check{A}$ , defined over  $K$ . We have a dual map  $\check{\phi} : \check{A} \rightarrow \check{A}$  and  $\check{\phi}$  kills the Cartier dual of  $\ker V_n$  which is  $\ker F^n$  on  $\check{A}$ . We can thus factor  $\check{\phi} = \psi \circ F^n, \psi : \check{A}^{(p^n)} \rightarrow \check{A}$ . We are done if  $\check{\phi}$  kills  $\ker[p]$ . But, if that is not the case there exists a cyclic subgroup  $H$  of  $\check{A}^{(p^n)}$  of order  $p^n$  on which  $\psi$  is injective. This subgroup will, moreover, be defined over  $K_s$ . Thus,  $\alpha(\psi(H))$  is a large subgroup of  $A$  of  $p$ -power order defined over  $K_s$ , which will be a contradiction for  $n$  sufficiently large.

One may conjecture, transposing a similar conjecture of Tate, that  $\text{End}(A) \otimes \mathbf{Z}_p$  is isomorphic to  $\text{End}_G(\Lambda)$ , if  $A$  is defined over a global field  $K$  with absolute Galois group  $G$  and  $A$  is sufficiently general. This is trivial if  $A$  is an elliptic curve, since both groups are isomorphic to  $\mathbf{Z}_p$  under the hypotheses. The first non-trivial case is when  $A$  is a product of two elliptic curves and in this case the conjecture is true, being essentially equivalent to Keating's characterization of the Igusa tower [Ke].

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Dept. of Mathematics, Univ. of Texas, Austin, TX 78712, USA

e-mail: voloch@math.utexas.edu