Anabelian geometry and descent obstructions on moduli spaces

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Abstract

Abstract: In this talk we study the section conjecture of anabelian geometry and the sufficiency of the finite descent obstruction to the Hasse principle for the moduli spaces of principally polarized abelian varieties and of curves over number fields. For the former we show that both the section conjecture and the finite descent obstruction fail in a very controlled way. For the latter, we prove some partial results that indicate that the finite descent obstruction suffices. We also show how this sufficiency implies the same for all hyperbolic curves.

Anabelian geometry

Grothendieck's anabelian program: The class of "anabelian" varieties can be recovered from their étale fundamental group plus Galois action of the absolute Galois group of the ground field.

Problem: Which varieties are "anabelian"?

Hyperbolic curves are anabelian. What else?

Fundamental group

X/K: smooth geometrically connected variety over a field K.

G_K: absolute Galois group of K

$\bar{X}$ base-change of X to an algebraic closure of K.

$\pi_1(.)$ the algebraic étale fundamental group functor on schemes (we omit base-points from notation).

Fundamental Exact Sequence

$$1 \to \pi_1(\bar{X}) \to \pi_1(X) \to G_K \to 1. \qquad (1)$$

Sections of fundamental exact sequence

$P \in X(K)$ gives section $G_K \to \pi_1(X)$ of the fundamental exact sequence, well-defined up to conjugation by $\pi_1(\bar{X})$. This defines $\sigma_{X/K} : X(K) \to H(K, X)$, the section map where

$H(K, X)$: set of sections $G_K \to \pi_1(X)$ modulo conjugation by $\pi_1(\bar{X})$

Section conjecture: $\sigma_{X/K}$ is a bijection if X is anabelian and K is finitely generated over its prime field.

Selmer sets and finite descent

K number field. K_v its completion at a place v.

Selmer set of X/K:

$H(K,X) \supset S(K,X) = \{s \mid \exists (P_v) \in \prod X(K_v), \alpha(s) = \prod \sigma_{X/K_v}(P_v)\}$.

$$
\begin{tikzcd}
 & X(K) \arrow[r] \arrow[d, "\sigma_{X/K}"'] & \prod X(K_v) \arrow[d, "\prod \sigma_{X/K_v}"] & \supset X^f \\
S(K,X) \subset & H(K,X) \arrow[r, "\alpha"] & \prod H(K_v, X).
\end{tikzcd}
$$

Related questions: When is $S(K,X) = \sigma_{X/K}(X(K))$? When is $X^f = X(K)$?

Moduli of abelian varieties

$\mathcal{A}_g$: Moduli space of principally polarized abelian varieties of dimension g.

$\pi_1(\bar{\mathcal{A}}_g) = Sp_{2g}(\hat{\mathbf{Z}})$ for $g > 1$. Non trivial, requires the congruence subgroup property for $Sp_{2g}(\mathbf{Z})$

$\sigma_{\mathcal{A}_g/K}(A) = \prod T_\ell(A)$, product of Tate modules.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\bar{\mathcal{A}}_g) \arrow[r] \arrow[d, "\cong"] & \pi_1(\mathcal{A}_g) \arrow[r] \arrow[d] & G_K \arrow[r] \arrow[d, "\chi"'] & 1 \\
1 \arrow[r] & Sp_{2g}(\hat{\mathbf{Z}}) \arrow[r] & GSp_{2g}(\hat{\mathbf{Z}}) \arrow[r] & \hat{\mathbf{Z}}^* \arrow[r] & 1.
\end{tikzcd}$$

Moduli of abelian varieties II

$\sigma_{\mathcal{A}_g/K}$ not injective. A pair of abelian varieties isogenous via two isogenies of coprime degrees have isomorphic Tate modules.

$\sigma_{\mathcal{A}_g/K}$ not surjective. Galois reps coming from abelian varieties have integral char polys of Frobenius.

Fontaine-Mazur conjecture. Well-behaved Galois reps occur as factors in twists of cohomology of algebraic varieties.

Effective motives conjecture (Katz, Mazur, Calegari). No twist needed if rep has integral char polys of Frobenius.

Moduli of abelian varieties III

Is $\sigma_{\mathcal{A}_g/K}(\mathcal{A}_g(K)) = S(K, \mathcal{A}_g)$?

Assuming the Fontaine-Mazur and Effective motives conjectures, an element of $S(K, \mathcal{A}_g)$ gives a Galois submodule of $\prod T_\ell(E)$ for an abelian variety E of dim. at least g. But not necessarily of dim. g.

For $g = 1$ and $K = \mathbf{Q}$ we get elliptic curve from Galois rep. using the (former) Serre conjecture (Helm, V.).

A counterexample

(Suggested by Zarhin) E/K simple abelian variety with $\mathrm{End}(E) \otimes \mathbf{Q} = D$, a quaternion algebra D over a number field F with $\dim E = 2[F : \mathbf{Q}] = 2g$. If K is big enough, and D splits at ℓ then $T_\ell(E)$ has a Galois submodule V_ℓ, which over K_v for v of good reduction for E comes from an abelian variety A_v/K_v of dim. g. As E is simple, the A_v do not globalize. I can make it work for all v and almost all ℓ or vice-versa but not both simultaneously.

Moduli of curves

$\mathcal{M}_g$: Moduli space of curves of genus g.

Conjecture: $\sigma_{\mathcal{M}_g/K}(\mathcal{M}_g(K)) = S(K, \mathcal{M}_g)$, K number field.

(If $\mathcal{M}_g$ anabelian, as conjectured by Grothendieck, then this follows)

Theorem (Mochizuki) $\sigma_{\mathcal{M}_g/K_v}$ is injective on $\mathcal{M}_g(K_v)$.

The conjecture plus Kodaira-Parshin imply that

$\sigma_{X/K}(X(K)) = S(K, X)$ bijectively for smooth, projective, hyperbolic curves.

THANK YOU

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