Planar surfaces in positive characteristic

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Abstract

A planar surface is a surface in three-space in which every tangent line has triple or higher contact with the surface at the point of tangency. We study properties of planar surfaces in positive characteristics, use that to bound the number of points of a planar surface over a finite field and give an application to Waring's problem for polynomials.

1 Introduction

Let S be a surface in three-space (for any reasonable interpretation of the term, for now). If P is a smooth point on S, then the lines that have at least double contact with S at P are the lines on the tangent plane to S at P that pass through P. These are called tangent lines to S at P. Typically, one expects that most tangent lines have contact exactly 2. A point P for which all tangent lines have contact at least 3 is called a planar point of S. A surface for which all smooth points are planar is called a planar surface. In characteristic zero, planar surfaces are planes. This is very classical in both the algebraic, analytic or differentiable settings. In positive characteristic this is no longer the case and now we restrict ourselves to the situation where we have an algebraic surface in $\mathbf{P}^3$ over a field of positive characteristic p. There are many more planar surfaces in positive characteristic, a classification of the smooth ones was obtained by Xu [Xu] who also proved that, on a planar surface, the generic order of contact at P of a tangent line with S is a power of p. We will give a somewhat different proof of these results under less restrictive hypotheses.

On a general point P of a non-planar surface, there are two (or occasionally just one) lines that touch S at a point with multiplicity at least three, called the asymptotic lines. The basic starting point of this paper is the observation (proved below) that, on a general point P of a planar surface, there is exactly one tangent line, which we call the hyperasymptotic line, which touches S at P with multiplicity higher than the general tangent line to S at P.

In this paper we will study various geometric questions connected to hyperasymptotic lines on planar surfaces. In addition, we give an application of our results to Waring's problem for polynomials.

In addition we will use them to give a bound on the number of rational points of the surface when the surface is defined over a finite field. This bound will be obtained by counting the number of points P of S in an algebraic closure of $\mathbf{F}_q$ whose image under the Frobenius map lies in the hyperasymptotic line to S at P.

A additional motivation for the study of planar surfaces is the paper of Ellenberg and Hablicsek [EH].

2 Geometry

In this section we fix an algebraically closed field k of characteristic $p > 0$. Let S be a closed, integral surface of degree $d > 1$ over k defined by the equation $f = 0$, where f is an irreducible homogeneous polynomial of degree d in $k[x_0, x_1, x_2, x_3]$. We assume that S is planar, that is, for all smooth points P on S, and all lines L tangent to S at P, the intersection multiplicity of the intersection of L and S at is at least 3. We say that S has controlled singularities if the one-dimensional component of the set of singular points of S is empty or is a curve of degree less than $d/2$ (compare [He]).

Theorem 1 *Let S be a planar surface with controlled singularities in characteristic $p > 0$ of degree d given by $f = 0$. Then there exists r, a power of p, such that $d \equiv 1 \pmod r$ and polynomials $g_i, i = 0, \ldots, 3$ such that $f = \sum g_i^r x_i$ and the intersection multiplicity of a general tangent line to S at a general point P is exactly r.*

Let C be a general plane section of S. From the assumption that S has controlled singularities, it follows that C has controlled singularities in the sense of [He]. Let P be a general point of C. The line to C at P is tangent to S at P with the same contact multiplicity and, since S is assumed planar, P is an inflection point of C. By the results of [He] (in particular Theorem 5.1), since P is general, there is r as in the statement of the theorem such that the contact of C (hence also S) with the tangent line at P is r and the equation for C is given by a similar expression (with one fewer variable) to that claimed for f. As C is a general plane section of S, it follows that f has the desired form.

Theorem 2 *On a general point P of a planar surface, there is exactly one tangent line which touches S at P with multiplicity higher than the general tangent line to S at P.*

Consider a general point P on the surface and choose affine coordinates x, y, z with P being at the origin and the tangent plane at P being $z = 0$. Expand z as a power series in x, y. Now, the condition that the generic order is r, gives that all partial derivatives of z of total order s with $1 < s < r$ vanish identically. It follows from that also that the mixed partials (i.e. those involving both x and y) of total order $s = r$ also vanish identically. Indeed, the mixed Hasse partial of order (i, j) is the composition of the i-th Hasse derivative with respect to x and the j-th Hasse derivative with respect to y (see [He], Section 3). If $i + j = r$ and both $i, j > 0$ then both $i, j < r$ and

$$D^{i,j}_{x,y} z = D^i_x (D^j_y z) = D^i_x(0) = 0$$

by the above, unless $j = 1$ (as $j < r$). If $j = 1$, then switch x and y. We are done, except if $i = j = 1$, i.e. $r = 2$ but S is not planar if $r = 2$. So the homogeneous term of order r of z is a r-th power of a linear form (as r is a power of the characteristic, by the previous theorem) and that linear form determines the unique tangent line with contact bigger than r. This completes the proof.

We will call the line, whose existence is asserted by the theorem, the hyperasymptotic line to S at P and denote it by L_P if the surface is fixed.

Note that the theorem does not have any assumptions on the singular locus of S, unlike the previous theorem. From the theorem it follows that a planar surface of degree r and generic order r is ruled, since the hyperasymptotic line at a general point has to be contained in S.

For a planar surface S of degree d and generic order r, given a point Q in $\mathbf{P}^3$, the equation of the surface, together with the first and r-th polars at Q, describe the set of points $P \in S$ with $Q \in L_P$. This is a system of equations of degree $d(d-1)(d-r)$ but since first polar is a r-th power, the expected number of points $P \in S, Q \in L_P$ is $d(d-1)(d-r)/r$.

On a planar surface S of degree d and generic order r, we consider the set X of points $P \in S$ for which the hyperasymptotic line to S at P has contact higher than $r+1$ or the generic tangent has contact higher than r. This is the analogue of the flecnodal locus and will will called the hyperflecnodal locus. One expects X to be a curve and one can bound its degree (following Kollar [K], 24) as follows. Recall, from theorem 1, that r is a power of p. We need to study the resultant of a system of forms of the shape

$$\sum a_i x_i = \sum b_i x_i^r = \sum c_{ij} x_i^r x_j = 0, i, j = 1, 2, 3$$

which has multidegree $(r^2+r, r+1, r)$ as can be seen by raising the first equation to the r-th power, solving the two linear equations in x_i^r and substituting in the r-th power of the third. Using

$$a_i = D_{x_i} f, b_i = D_{x_i}^r f, c_{ij} = D_{x_i}^r D_{x_j} f,$$

leads to an equation cutting X inside S of degree at most

$$(r^2+r)(d-1) + (r+1)(d-r) + r(d-r-1) = (r^2+3r+1)d - 3r(r+1).$$

This argument fails if every point on S is hyperflecnodal, that is, the order of contact of the hyperasymptotic line at a generic point $P \in S$ with S at P is always bigger than $r+1$. In this case, unless S is ruled, there exists a finite $s > r+1$ which is the order of contact of the hyperasymptotic line at a generic point $P \in S$ with S at P. Now a modification of the above argument leads to a resultant of equations of multidegree (rs, s, r), giving an equation cutting X inside of S of degree at most $d(sr(d-1) + s(d-r) + r(d-s))$ and thus an upper bound of $d(sr(d-1) + s(d-r) + r(d-s))$ for the degree of X. In particular, it follows that S contains at most $d^2(d^2-2d+2)$ lines (compare [K], prop. 26). An interesting question is to determine the possible values for r, s given d, p. Clearly $r < s \leq d$. A guess would be that $s = r+1, 2r$ or another power of p.

3 Arithmetic

Consider a planar surface S of degree d and generic order r, defined over a finite field of cardinality q, such that not every point of S is hyperflecnodal.

We will give a bound on the number of rational points of the surface using the hyperasymptotic lines. This bound will be obtained by counting the number of points P of S in an algebraic closure of $\mathbf{F}_q$ whose image under the Frobenius map lies in the hyperasymptotic line to S at P.

Theorem 3 *Let S be planar surface of degree d and generic order r, defined over a finite field of cardinality q, such that not every point of S is hyperflecnodal. Then*

$$\#S(\mathbf{F}_q) \leq \frac{d(d+q-1)(d+rq-r)}{r(r+1)} + d((r^2+3r+1)d - 3r(r+1))(q+1).$$

The proof is similar to theorem 2 of [V]. Consider the algebraic set Z of points P of S in an algebraic closure of $\mathbf{F}_q$ whose image under the Frobenius map lies in the hyperasymptotic line to S at P. The expected number of points satisfying this condition is $d(d+q-1)(d+qr-r)$ with the rational points being counted with multiplicity at least $r(r+1)$, with equality if the rational point is not hyperflecnodal (by a calculation similar to that of [V]). So, if Z has a one-dimensional component, it is contained in the hyperflecnodal locus of S, which has degree at most $d((r^2+3r+1)d - 3r(r+1))$, as we saw in the previous section. The first term of the inequality of the theorem is an upper bound on the isolated points of Z and the second accounts for the other points.

4 Examples

The Hermitian surface $S : x^{p+1} + y^{p+1} + z^{p+1} = 1$ is planar in characteristic p. If F_2 denotes the $\mathbf{F}_{p^2}$-Frobenius then the line $PF_2(P)$ is tangent to S at P if $P \neq F_2(P)$. Since the order of contact is at least p, then it must be exactly p, unless the line is entirely contained in S, as the surface has degree $p+1$. Also, if a line has contact bigger than $p+1$ at a point then the line is contained in S. S contains $(p+1)(p^3+1)$ lines and $(p^2+1)(p^3+1)$ $\mathbf{F}_{p^2}$-rational points with $p+1$ lines passing through each of them ([Hi] Thm 19.1.5). The locus $\{P \in S | F_2(P) \in L_P\}$ consists of exactly the lines contained in S which is also the locus of points for which the hyperasymptotic line has contact bigger than $p+1$. If $q = p^m, m > 2$, the set Z in the proof of the theorem of the previous section contains, in addition to the lines, the set of $\mathbf{F}_q$ and $\mathbf{F}_{q/p^2}$ rational points.

The surface $z = (x^6 + y^6 + 1)y$, $p = 3$ is planar ([Xu]). It contains three lines $z = y = 0$ and $z = y, x = \pm iy$, where $i^2 = -1$. Except at the origin, the generic tangent line has contact 3 and the hyperasymptotic line has contact at least 6, with equality outside of the lines.

We now consider the surface $X : x^{2q+1} + y^{2q+1} + z^{2q+1} = 1$, where q is a power of the characteristic p. Note that it has automorphisms consisting of permuting the variables (projectively, even) and multiplying some variables by $2q+1$-st roots of unity. In what follows, we will often work in an affine piece of X and use the automorphisms to extend the results to the whole of X. Note that $z = \zeta, \zeta^{2q+1} = 1$ meets X in $2q+1$ lines and, using the automorphisms, we get $3(2q+1)^2$ lines in X.

We begin by computing the hyperasymptotic line at a general point of X and an equation for the hyperflecnodal locus. Consider a line $(x+t, y+at, z+bt)$ through $(x, y, z) \in X$.

$$(x+t)^{2q+1}+(y+at)^{2q+1}+(z+bt)^{2q+1}-1 = C_1t+C_2t^q+C_3t^{q+1}+C_4t^{2q}+C_5t^{2q+1}.$$

The conditions $C_1 = C_2 = 0$ impose linear relations on a^q, b^q and define the hyperasymptotic line. Explicitly,

$$C_1^q = x^{2q^2} + y^{2q^2}a^q + z^{2q^2}b^q$$

and

$$C_2 = x^{q+1} + y^{q+1}a^q + z^{q+1}b^q$$

so the conditions are

$$\begin{aligned} a^q &= -(x^{2q^2}z^{q+1} - z^{2q^2}x^{q+1})/(y^{2q^2}z^{q+1} - z^{2q^2}y^{q+1}), \\ b^q &= -(x^{2q^2}y^{q+1} - y^{2q^2}x^{q+1})/(z^{2q^2}y^{q+1} - y^{2q^2}z^{q+1}). \end{aligned}$$

The equation for the hyperflecnodal locus is obtained by substituting these values of a^q, b^q into

$$C_3^q = x^{q^2} + y^{q^2}a^{q(q+1)} + z^{q^2}b^{q(q+1)}.$$

yielding

$$x^{q^2}(y^{2q^2}z^{q+1}-z^{2q^2}y^{q+1})^{q+1}+y^{q^2}(x^{2q^2}z^{q+1}-z^{2q^2}x^{q+1})^{q+1}+z^{q^2}(x^{2q^2}y^{q+1}-y^{2q^2}x^{q+1})^{q+1}$$

If we abbreviate $u = x^{2q+1}, v = y^{2q+1}, w = z^{2q+1}$, the above expression becomes $(xyz)^{q^2} F$, where

$$F = uv(u^{q-1} - v^{q-1})^{q+1} + uw(u^{q-1} - w^{q-1})^{q+1} + vw(v^{q-1} - w^{q-1})^{q+1}.$$

So, the hyperflecnodal locus consists of the curves given by $xyz = 0$ as well as the curve D described by the equations $F = 0, u + v + w = 1$. The hyperosculating line at a general point of $xyz = 0$ has contact $2q + 1$ with the point. For instance, if $z = 0$, the line is given by (x, y, t).

Note that D contains the set of lines given by $(u+v)(u+w)(v+w) = 0$. We proceed to find the singular locus of the curve $F = 0, u + v + w = 1$ in the variables u, v, w. The singular locus is described by the extra equations $\partial F/\partial u = \partial F/\partial v = \partial F/\partial w = 0$. Since F is homogeneous, this forces all three partials to vanish if $p > 3$, which we assume henceforth. Now,

$$\begin{aligned} \partial F/\partial u = v(u^{q-1} - v^{q-1})^{q+1} - uv(u^{q-1} - v^{q-1})^q u^{q-2} + \\ w(u^{q-1} - w^{q-1})^{q+1} - uw(u^{q-1} - w^{q-1})^q u^{q-2} = \\ -(v(u^{q-1} - v^{q-1}) + w(u^{q-1} - w^{q-1}))^q \end{aligned}$$

which simplifies to $-(u^{q-1} - 1)^q$ upon $w = 1 - u - v$. In particular, the singular locus of curve $F = 0, u + v + w = 1$ consists of the points with $u^{q-1} = v^{q-1} = w^{q-1} = 1$. We remark that, in particular, this curve is reduced. We use this to show that, on a general point of the hyperflecnodal locus, we have $C_4 \neq 0$, i.e., the order of contact of the hyperasymptotic line with the point is exactly $2q$. To see this, note that $C_4 = x + ya^{2q} + zb^{2q}$ and, substituting the above expressions for a^q, b^q in this, transforms the condition $C_4 = 0$ into $(xyz)^2 G = 0$, where

$$G = uv(u^{q-1} - v^{q-1})^2 + uw(u^{q-1} - w^{q-1})^2 + vw(v^{q-1} - w^{q-1})^2.$$

Now, the equation $F = 0$ defines a reduced curve of degree $q^2 + 1$ in the plane $u + v + w = 1$, while the equation $G = 0$ defines a curve of degree $2q$, so the first curve cannot be a subset of the second, by degree considerations. Note that $G = 0$ also contains $(u + v)(u + w)(v + w) = 0$, so they do have components in common.

5 Waring's problem

Let R be a ring and $n > 1$ a fixed integer. Waring's problem in this setting is to determine the least integer s for which every element of R is a sum of s n-th powers of elements of R, if such an integer exists, or ∞ otherwise. Note that what is usually called Waring's problem is not what we call Waring's problem for $\mathbf{Z}$. For n odd, what we call Waring's problem for $\mathbf{Z}$ is usually referred to as the "easier" Waring's problem, with Waring's problem proper referring only to positive integers. In this section, we consider Waring's problem for $R = k[t]$, where k is an algebraically closed field of characteristic p and we denote the least s as above by $v(p, n)$. This problem has been extensively studied ([C, LW] and references therein). For $p = 0$, it's known that $\sqrt{n} < v(0, n) \leq n$ ([NS]). Our focus here is on $p > 0$. If $n = k_0 + k_1 p + \cdots + k_r p^r$ is the base p expansion of n (i.e. $0 \leq k_i < p$), then Liu and Wooley [LW] showed that $v(p, n) \leq \prod(k_i + 1)$. We improve this bound for some values of n and relate this to the geometric discussion above.

Note that, if s is the smallest integer for which there exists $x_1, \ldots, x_s \in k[t]$ with $\sum x_i^n = t$, then $s = v(p, n)$, simply by replacing t by a polynomial in t. It is easy to see that $v(p, 2) = 2, p > 2$, that $v(p, n) > 2$ for all $n > 2$ and that $v(p, n) = \infty$ if $p | n$. The following proposition for $n = p^m + 1$ is due to Car, [C], Prop. 3.2. Our proof, although similar, gives slightly more information and motivates our next result.

Proposition 1 *If $n | (p^m + 1)$ for some m, then $v(p, n) = 3$.*

Let us write $q = p^m$. An identity $\sum x_i^{q+1} = t$ gives $\sum (x_i^{(q+1)/n})^n = t$, so we need only consider $n = q + 1$. Let $x, y \in k$ satisfy $x^{q+1} + y^{q+1} + 1 = 0$, then

$$(xt + x^{q^2})^{q+1} + (yt + y^{q^2})^{q+1} + (t + 1)^{q+1} = ct,$$

where $c = x^{q^3+1} + y^{q^3+1} + 1$ and can be chosen to be nonzero by an appropriate choice of x, y. Replacing t by t/c completes the proof.

We remark that the solutions to $x^{q^3+1} + y^{q^3+1} + 1 = x^{q+1} + y^{q+1} + 1 = 0$ are in $\mathbf{F}_{q^2}$.

The identity $\sum_{i=1}^{s} (x_i(t))^n = t$, for polynomials $x_i(t) \in k[t]$ can be interpreted as follows. The rational curve parametrized by $(x_1(t) : \ldots : x_s(t))$ in $\mathbf{P}^{s-1}$ meets the hypersurface X given by the equation $\sum_{i=1}^{s} x_i^n = 0$ at $t = 0$ with multiplicity one and at $t = \infty$ with multiplicity $n(\max \deg x_i) - 1$ (the latter because there is no intersection with $t \neq 0, \infty$ and Bézout). The identity from the previous proposition follows from the fact that the Hermitian curve $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0$ is Frobenius non-classical in the sense of [SV], more precisely, the tangent line at general point P of the curve has contact q with the curve and also contains the image of P under the q^2-Frobenius map. It follows from the known properties of the Fermat curves $x_1^n + x_2^n + x_3^n = 0$ that there are no linear polynomials $x_i(t) \in k[t]$ with $\sum_{i=1}^{3}(x_i(t))^n = t$, unless n is of the form $q+1$. We conjecture that $v(p,n) > 3$ in the cases not covered by the above proposition.

Theorem 4 *If $p > 3$ and $n|(2p^m+1)$ for some m, then $v(p,n) \leq 4$.*

As in the previous proof, we reduce to the case $n = q+1$. From the example in the previous section, the surface $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} + x_4^{q+1} = 0$ has points (viz. the general point on the hyperflecnodal locus) for which there is a line that intersects the surface at the point with multiplicity exactly $2q$. Parametrizing such a line in a way that the point of tangency correspond to $t = \infty$ and the additional point of intersection corresponds to $t = 0$ proves the proposition from the discussion preceding it.

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