SIEGEL'S THEOREM FOR COMPLEX FUNCTION FIELDS

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Abstract. We give a short proof of the finiteness of the set of integral points on an affine algebraic curve of genus at least one, defined over a function field of characteristic zero.

Siegel [Si] has shown that an affine algebraic curve of genus at least one defined over a number field, has only finitely many integral points. Lang [L] has proven an analogous result for curves defined over a function field of characteristic zero, not defined over the constant field. For curves of genus at least two, one even has the Mordell conjecture (proved by Faltings [F] in the number field case and by Manin [M] in the function field case) that there are only finitely many rational points.

For genus one, Manin [M] gave a proof of a strenghtening of Lang's result as a by-product of his work on the Mordell conjecture. Mason [Ms] then gave an effective proof by more elementary considerations. In this note, we give a short proof of Manin's (and hence Lang's) result for genus one. The proof can be adapted to higher genus as well (see the remark below).

Let K be a function field with constant field of characteristic zero and E/K an elliptic curve with non-constant j-invariant. The reader may consult [S] for definitions and results about elliptic curves. In particular we shall use the following results. The group $E(K)$ is a finitely generated abelian group by the Mordell-Weil theorem and there is a height function $h : E(K) \to \mathbf{R}$ with the property that there are only finitely many points of bounded height ([L], Proposition 2). The height can be written as a sum of local heights $\sum \lambda_v(P)$, where v ranges through the places of K. The local heights satisfy $\lambda_v(P) = max\{0, v(t(P))\} + \beta_v(P)$, where β_v is bounded for all v and it is identically zero for all but finitely many v, and t is a uniformizer at $0 \in E$. For example, $t = x/y$, where x, y are coordinates of a Weierstrass equation for E.

Now, Lang's result for genus one can be reduced to the case of a Weierstrass equation ([S], Corollary IX.3.2.2) and in this case we argue as follows. Let S be a finite set of places of K. Then $\sum_{v \notin S} \lambda_v(P)$ is bounded independently of P, if P is S-integral and, since there are only finitely many points of bounded height, if there are infinitely many S-integral points, then λ_v is unbounded for some $v \in S$. It suffices thus to prove the following result:

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Theorem (Manin). *Let K be a function field with constant field of characteristic zero and E/K an elliptic curve with non-constant j-invariant and v a place of K. Then the local height function λ_v is bounded on $E(K)$.*

Proof. The points on $E(K_v)$ that reduce to 0 mod v form a subgroup $E_1(K_v)$ isomorphic to the group of points of a formal group. Choosing an uniformizer t, as above, on E at 0, then $E_1(K_v) = \{P \in E(K_v) | t(P) \in \mathcal{M}_v\}$, where $\mathcal{M}_v$ is the maximal ideal of the local ring at v. Moreover, $\lambda_v(P)$ differs from $v(t(P))$ by a bounded amount. Hence, it suffices to show that $v(t(P))$ is bounded above on $E(K)$. Suppose not and choose $P_n, n = 1, 2, \ldots$ in $E(K)$ such that $v(t(P_{n+1})) > v(t(P_n)) > 0$. We claim that $P_1, P_2, \ldots$ are linearly independent over $\mathbf{Z}$. Recall that t induces a group isomorphism between E_r/E_{r+1}, where $E_r = E_r(K_v) = \{P \in E(K_v) | t(P) \in \mathcal{M}_v^r\}$, and $\mathcal{M}_v^r/\mathcal{M}_v^{r+1}$. If $n_i P_i = \sum_{j>i} n_j P_j, n_i \neq 0$ and $r = v(t(P_i))$ then $n_i P_i$ is 0 in E_r/E_{r+1}, but $t(n_i P_i) \equiv n_i t(P_i) \not\equiv 0 \pmod{\mathcal{M}_v^{r+1}}$, which proves the claim. On the other hand, the claim contradicts the Mordell-Weil theorem and this completes the proof.

Remark. On a curve of genus greater than one, if a sequence of points $P_1, P_2, \ldots$ approach rapidly a point P_∞, then a similar argument shows that the $P_i - P_\infty$ are linearly independent over $\mathbf{Z}$ in the Jacobian of the curve, and Lang's result follows from this. The author and A. Buium [BV] have recently proved a conjecture of Lang to the effect that an affine open subset of an abelian variety of any dimension over a function field of characteristic zero has finitely many integral points.

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