UNBOUNDEDNESS OF THE NUMBER OF RATIONAL POINTS ON CURVES OVER FUNCTION FIELDS

RICARDO CONCEIÇÃO, DOUGLAS ULMER, AND JOSÉ FELIPE VOLOCH

ABSTRACT. We give examples of sequences of smooth non-isotrivial curves for every genus at least two, defined over a rational function field of positive characteristic, such that the (finite) number of rational points of the curves in the sequence cannot be uniformly bounded.

The question of whether there is a uniform bound for the number of rational points on curves of fixed genus greater than one over a fixed number field has been considered by several authors. In particular, Caporaso et al. [CHM97] showed that this would follow from the Bombieri-Lang conjecture that the set of rational points on a variety of general type over a number field is not Zariski dense. Abramovich and the third author [AV96] extended this result to get other uniform boundedness consequences of the Bombieri-Lang conjecture and gave some counterexamples for function fields. These counterexamples are singular curves that "change genus". They behave like positive genus curves (and, in particular, have finitely many rational points), but are parametrizable over an inseparable extension of the ground field. In [AV96] it is shown that, for this class of equation, uniform boundedness does not hold. Specifically, one gets a one-parameter family of equations which, for suitable choice of the parameter, have a finite but arbitrarily large number of solutions. However, a negative answer to the original uniform boundedness question for smooth curves of genus at least two remained open in the function field case. In this paper we provide counterexamples to this uniform boundedness, extending constructions of the first two authors [Con09, Ulm09] for elliptic curves.

Theorem. *Let $p > 3$ be a prime number and let r be an odd number coprime to p. The number of rational points over $\mathbf{F}_p(t)$ of the curve X_a with equation $y^2 = x(x^r + 1)(x^r + a^r)$ is unbounded as a varies in $\mathbf{F}_p(t) \setminus \mathbf{F}_p$.*

Proof. We first note that if $d = p^n + 1$ and $a = t^d$, then we have a rational point $(x, y) = (t, t^{(r+1)/2}(t^r + 1)^{d/2})$ on X_a. Second, if m divides

Date: April 9, 2012.

n and n/m is odd, then $d' = p^m + 1$ divides d. Setting $e = d/d'$, we have another rational point $(x, y) = (t^e, t^{e(r+1)/2}(t^{re} + 1)^{d'/2})$ on X_a. Thus if we take n to be odd with many factors, we have many points. □

The curves given by the theorem are non-isotrivial of odd genus r. To obtain counterexamples of even genus, one may proceed as follows: Let Y_a be the quotient of X_a by the fixed-point-free involution $(x, y) \mapsto (a/x, -a^{(r+1)/2}y/x^{r+1})$. Then X_a is an unramified cover of Y_a, a curve of genus $(r + 1)/2$. This shows that for all $g > 1$ and all but finitely many $p > 2$ (the exceptions depending on g), unboundedness of the number of rational points over $\mathbf{F}_p(t)$ holds for curves of genus g.

One can obtain other explicit examples by a slight modification of the argument of the theorem. In what follows $d = p^n + 1$, $m|n$ with n/m odd, $d' = p^m + 1$, $e = d/d'$, and $a = t^d$, as in the proof of the theorem.

Example 1. Let $p \equiv 2 \bmod 9$ and n be divisible by 3 and a product of primes $\equiv 1 \bmod 6$. Then the curve $y^6 = x(x + 1)(x + a)$ contains the point $(t^e, t^e(t^e + 1)^{d'/2})$.

Example 2. Let $f(x) \in \mathbf{F}_p[x]$ be a polynomial of degree $2b$ with distinct roots, none of them zero. Then the curve $y^2 = f(x)x^{2b}f(a/x)$ has the point $(t^e, t^{be}f(t^e)^{d'/2})$.

Example 3. Let r be a prime satisfying $p \equiv r - 1 \bmod r^2$. Let n be divisible by r and primes $\equiv 1 \bmod r(r - 1)$. Then $(t^e, t^{2e/r}(t^e + 1)^{d'/r})$ is a point on the curve $y^r = x(x + 1)(x + a)$. This curve has a simple Jacobian, since the ring of integers of the r-th cyclotomic field acts as endomorphisms of the Jacobian (see [Zar06], Theorem 3.1).

Remark. The curve in Example 1 and the curve in the theorem cover the Legendre elliptic curve E studied in [Ulm09]. Thus their Jacobians have E as a factor, and consequently have unbounded Mordell-Weil rank as a varies. On the other hand, E is not a factor of the Jacobian of the curve Y_a, nor of the Jacobians of the curves in Examples 2 and 3. Nonetheless, by the main result of [BV96], the rank of the Mordell-Weil group of the Jacobian of these curves is also unbounded as a varies.

Let X be the smooth projective surface with affine model

$$y^2 = x(x^r + 1)(x^r + t^r).$$

By [CHM97] the fibration $X \to \mathbf{P}^1$, $(x, y, t) \mapsto t$ has a fibered power which covers a variety of general type. However, since this fibration is defined over a finite field, this variety of general type will also be defined over a finite field and can have a Zariski dense set of $\mathbf{F}_p(t)$-rational points, so the rest of the argument of [CHM97] does not apply.

Acknowledgments: We would like to thank Bjorn Poonen for his suggestions, in particular for suggesting the involution that leads to the curves Y_a.

References

[AV96] D. Abramovich and J. F. Voloch. Lang's conjectures, fibered powers, and uniformity. *New York J. Math.*, 2:20–34, electronic, 1996.

[BV96] A. Buium and J. F. Voloch. Lang's conjecture in characteristic p: an explicit bound. *Compositio Math.*, 103:1–6, 1996.

[Con09] R. Conceição. *Twists of Elliptic curves with a large set of integral points over* $\mathbf{F}_q(t)$. 2009. Thesis (Ph.D.)–The University of Texas at Austin, arXiv:0910.3417.

[CHM97] L. Caporaso, J. Harris, and B. Mazur. Uniformity of rational points. *J. Amer. Math. Soc.*, 10:1–35, 1997.

[Ulm09] D. Ulmer. Explicit points on the Legendre curve. 2009. Preprint, arXiv:1002.3313.

[Zar06] Yu. G. Zarhin. Superelliptic jacobians. In: Diophantine Geometry Proceedings (U. Zannier, ed.), Edizioni Della Normali, Pisa, pp. 363-390 (2007)

Department of Mathematics, Oxford College of Emory University, Oxford, GA 30054, USA
E-mail address: rconcei@emory.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA
E-mail address: ulmer@math.gatech.edu

Department of Mathematics, University of Texas, Austin, TX 78712, USA
E-mail address: voloch@math.utexas.edu