On the Klein–Kroll types of flat Minkowski planes *

Günter F. Steinke

Abstract

Klein and Kroll classified Minkowski planes with respect to subgroups of central automorphisms and obtained a multitude of types. Some of these types are known to exist only in finite Minkowski planes or as the type of a proper subgroup in the automorphism group. In this paper we investigate the Klein–Kroll types of flat Minkowski planes with respect to the full automorphism groups of these planes and completely determine all possible types of flat Minkowski planes. For each of these 14 types except type II.A.15 examples are given.

Mathematics Subject Classification (2000): 51H15, 51B20.
Key words: flat Minkowski plane, central automorphism, Klein–Kroll type

1 Introduction and Preliminaries

Similar to the Lenz–Barlotti classification of projective planes with respect to central collineations and the Hering classification of Möbius planes with respect to central automorphisms, Minkowski planes can be classified with respect to *central automorphisms*, that is, permutations of the point set such that parallel classes are mapped to parallel classes and circles are mapped to circles and such that at least one point is fixed and central collineations in the derived projective plane at each fixed point are induced. More precisely, one considers subgroups of central automorphisms which are linearly transitive, that is, the induced groups of central collineations are transitive on each central line except for the obvious fixed points, the centre and the point of intersection with the axis.

Unlike in the case of Möbius planes, where we have only three types of central automorphisms, central automorphisms of Minkowski planes come in a variety of types according to whether the axis of a central collineation in the derived projective plane at a fixed point is the line at infinity, a line that stems from a circle of the Minkowski plane or a line that comes from a parallel class of the Minkowski plane, and whether or not the centre is on the axis of the central collineation.

*This paper was published in J. Geom. 87 (2007), 160–178, DOI 10.1007/s00022-007-1913-z

Klein and Kroll therefore obtained far more types of Minkowski planes. They considered three types of central automorphisms, q-translations, G-translations and (p,q)-homotheties, see the following sections. In fact, in their classification they considered groups of automorphisms and determined their types according to transitive subgroups of central automorphisms contained in them. In the following we only deal with the full automorphism group. We furthermore say that a flat Minkowski plane $\mathcal{M}$ is of Klein–Kroll type X if the full automorphism group of $\mathcal{M}$ is of type X, see the following sections for the definitions of the various Klein–Kroll types.

In this paper we completely determine which Klein-Kroll types can occur in flat Minkowski planes except for one type which seems perfectly feasible in flat Minkowski planes but no examples are known so far. We further characterise the Artzy-Groh planes and the planes admitting a 3-dimensional kernel in terms of their Klein-Kroll types, see sections 2.2 and 2.1 for these kinds of Minkowski planes.

In Section 2 we review the basic theory of flat Minkowski planes and provide descriptions of certain families of such planes. The next three sections deal with q-translations, G-translations and (p,q)-homotheties, respectively, and determine in each case the possible Klein-Kroll types with respect this kind of central automorphisms. At the end of Section 5 the classification with respect to all three types of central automorphisms combined is presented and characterisations of some well known families of flat Minkowski planes in terms of their Klein-Kroll types are given. Finally, the last section provides examples for each of the combined types bar one.

2 Preliminaries and Some Families of Flat Minkowski Planes

In this section we collect more information about flat Minkowski planes and we present, for easy reference, brief descriptions and characterisations of certain flat Minkowski planes which we shall refer to in the following sections. For more details about these families of Minkowski planes we refer to the relevant papers or [11] Chapter 4; for an algebraic treatment of Minkowski planes see [2].

A *Minkowski plane* $\mathcal{M} = (P, \mathcal{C}, \{||_+, ||_-\})$ is an incidence structure consisting of a point set P, a circle set $\mathcal{C}$, elements of which are non-empty subsets of P, and two equivalence relations $||_+$ and $||_-$ (parallelisms) defined on the point set such that three mutually non-parallel points can be joined by a unique circle, such that the circles which touch a fixed circle K at $p \in K$ partition $P \setminus (|p|_+ \cup |p|_-)$ where $|p|_\pm$ denotes the $(\pm)$-parallel class of p, such that each parallel class meets each circle in a unique point (parallel projection), such that each $(+)$-parallel class and each $(-)$-parallel class intersect in a unique point, and such that there is a circle that contains at least three points (richness); compare [13]. It readily follows that for each point p of $\mathcal{M}$ the incidence structure $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ whose point set A_p consists of all points of $\mathcal{M}$ that are not parallel to p and whose line set $\mathcal{L}_p$ consists of all restrictions to A_p of circles of $\mathcal{M}$ passing through p and of all parallel classes not passing through p is an affine plane, called the *derived affine plane at p*. This affine plane extends to a projective plane $\mathcal{P}_p$, which we call the *derived projective plane at p*.

In this paper we are only concerned with *flat Minkowski planes* whose common point set is the torus $\mathbb{S}^1 \times \mathbb{S}^1$ (where $\mathbb{S}^1$ usually is represented as $\mathbb{R} \cup \{\infty\}$), whose circles are graphs of homeomorphisms of $\mathbb{S}^1$ and whose parallel classes of points are the horizontals and verticals on the torus (standard representation). In these planes the geometric operations of joining three mutually non-parallel points by a circle, intersecting of two circles, and touching become continuous when the circle set is suitably topologised. The flat Minkowski planes are precisely the *2-dimensional topological Minkowski planes* in the sense of [13]. For more information on flat Minkowski planes we refer to [13] and [11] Chapter 4.

An *automorphism* of a Minkowski plane is a permutation of the point set such that parallel classes are mapped to parallel classes and circles are mapped to circles. Every automorphism of a flat Minkowski plane is continuous and thus a homeomorphism of the torus. The collection of all automorphisms of a 2-dimensional Minkowski plane $\mathcal{M}$ forms a group with respect to composition, the automorphism group Γ of $\mathcal{M}$. This group is a Lie group of dimension at most 6 with respect to the compact-open topology; see [11] Theorem 4.4.4. We call the dimension of Γ the *group dimension* of $\mathcal{M}$. The collection of all automorphisms of $\mathcal{M}$ that fix each $(\pm)$-parallel class is a closed normal subgroup of Γ, called the *kernel $T^\pm$* of $\mathcal{M}$. Furthermore $T^\pm$ is at most 3-dimensional.

All flat Minkowski planes of group dimension at least 4 or kernel dimension 3 have been classified by Schenkel [13], see also [11] section 4.4, or Theorems 2.3 and 2.2 below.

2.1 Swapping Halves and the Planes $\mathcal{M}(f, g)$

Let $\mathcal{M} = (\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C})$ be a flat Minkowski plane in standard representation. Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the sets of all circles in $\mathcal{C}$ that are graphs of orientation preserving and orientation-reversing homeomorphisms $\mathbb{S}^1 \to \mathbb{S}^1$, respectively. Clearly, $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$. We call $\mathcal{C}^+$ and $\mathcal{C}^-$ the *positive* and *negative half of $\mathcal{M}$*, respectively. These two halves are completely independent of each other, that is, we can combine these parts from different flat Minkowski planes and obtain another flat Minkowski plane; see [11].

THEOREM 2.1 *Let $\mathcal{M}_i = (\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C}_i)$, $i = 1, 2$ be two flat Minkowski planes and let $\mathcal{C} = \mathcal{C}_1^+ \cup \mathcal{C}_2^-$. Then the geometry $\mathcal{M} = (\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C})$ is a flat Minkowski* *plane.*

We say that $\mathcal{M}$ as in the theorem above is obtained from $\mathcal{M}_1$ by swapping the negative half with $\mathcal{M}_2$. Note that interchanging the roles of $\mathcal{M}_1$ and $\mathcal{M}_2$ yields another Minkowski plane, which is obtained from $\mathcal{M}_1$ by swapping the positive half with $\mathcal{M}_2$.

For example, this method can be used to construct a variety of Minkowski planes from two different models of the classical flat Minkowski plane. In order to do this we denote by Ξ the projective linear group $\mathrm{PGL}(2,\mathbb{R})$ of all fractional linear transformations of $\mathbb{S}^1$—that is, the transformations

$$x \mapsto \frac{ax+b}{cx+d}$$

for $a,b,c,d \in \mathbb{R}$, $ad-bc \neq 0$, with the obvious definitions for $x=\infty$ and when the denominator becomes 0, that is, the convention $\frac{1}{\infty}=0$, $\frac{1}{0}=\infty$ and $a\cdot\infty+b=\infty$ for all $a,b \in \mathbb{R}$, $a \neq 0$, are used—and by Λ the normal subgroup $\mathrm{PSL}(2,\mathbb{R})$ of index 2 in Ξ consisting of all orientation preserving transformations in Ξ. Then circles in the positive and negative half of the *classical flat Minkowski plane* $\mathcal{M}_1$ are the graphs of transformations in Λ and $\Xi \setminus \Lambda$, respectively.

Let f and g be two orientation preserving homeomorphisms of $\mathbb{S}^1$. The graphs of homeomorphisms in $g^{-1}\Xi f$ yields another copy $\mathcal{M}_2$ of the classical flat Minkowski plane. We now form the Minkowski plane $\mathcal{M}(f,g)$ obtained from $\mathcal{M}_1$ by swapping the negative half with $\mathcal{M}_2$, that is, $\mathcal{M}(f,g)$ has point set the torus $\mathbb{S}^1\times\mathbb{S}^1$, parallel classes are of the form $\{x_0\}\times\mathbb{S}^1$ and $\mathbb{S}^1\times\{y_0\}$ for $x_0,y_0\in\mathbb{S}^1$ and circles are of the form $\{(x,\gamma(x))|\ x\in\mathbb{S}^1\}$ for $\gamma\in\Lambda\cup g^{-1}(\Xi\setminus\Lambda)f$.

Note that each flat Minkowski plane $\mathcal{M}(f,g)$ is isomorphic to one where both describing homeomorphisms f and g fix ∞, 1 and 0 and that the homeomorphism $(x,y)\mapsto(y,x)$ of $\mathbb{S}^1\times\mathbb{S}^1$ to itself defines an isomorphism from $\mathcal{M}(f,g)$ to $\mathcal{M}(g,f)$.

For the classification of these planes with respect to their group dimensions see [15], Theorem 3.9, or [11], Theorem 4.3.4, and with respect to their Klein–Kroll types see [16] 6.2. The maximum dimension of a kernel is attained precisely in the flat Minkowski planes $\mathcal{M}(f,id)$; cf. [13].

THEOREM 2.2 *A flat Minkowski plane $\mathcal{M}$ is isomorphic the a plane $\mathcal{M}(f,id)$ if and only if one of the following is satisfied:*

- *one of the kernels $T^\pm$ of $\mathcal{M}$ is 3-dimensional;*
- *one of the kernels $T^\pm$ of $\mathcal{M}$ contains a subgroup isomorphic to* $\mathrm{PSL}(2,\mathbb{R})$.

Furthermore, a flat Minkowski plane is classical if and only if one of the kernels $T^\pm$ is 3-dimensional and at least one derived affine plane is Desarguesian.

2.2 Artzy–Groh Planes and Generalised Hartmann Planes

Let $f, g : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ be two homeomorphisms. For $a, b, c \in \mathbb{R}$ let

$$C_{a,b,c} = \begin{cases} \{(x, bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\} & \text{for } a = 0, \\ \{(x, af(\frac{x-b}{a}) + c) \mid x \in \mathbb{R}\} \cup \{(\infty, c), (b, \infty)\} & \text{for } a > 0, \\ \{(x, ag(\frac{x-b}{|a|}) + c) \mid x \in \mathbb{R}\} \cup \{(\infty, c), (b, \infty)\} & \text{for } a < 0, \end{cases}$$

and let

$$\mathcal{C}_{f,g} = \{C_{a,b,c} \mid a, b, c \in \mathbb{R}\}.$$

Artzy and Groh showed in [1] Theorem 4.8 that a flat Minkowski plane that admits the group $\{(x, z) \mapsto (rx + b, ry + c) \mid b, c, r \in \mathbb{R}, r > 0\}$ as a group of automorphisms is isomorphic to a plane whose circle set is $\mathcal{C}_{f,g}$ where f and g are homeomorphisms $\mathbb{S}^1 \to \mathbb{S}^1$ whose restrictions to $\mathbb{R} \setminus \{0\}$ are differentiable and for 0, 1 and ∞ take on the values $f(\infty) = g(\infty) = 0$, $f(0) = g(0) = \infty$, $f(1) = g(1) = 1$. We denote this plane by $\mathcal{M}^{AG}(f, g)$ in order to distinguish it from a plane admitting a 3-dimensional kernel from the previous section.

A *semi-multiplicative homeomorphism* of $\mathbb{S}^1$ is a homeomorphism f of the form

$$f(x) = f_{r,s}(x) = \begin{cases} x^r, & \text{if } x \in \mathbb{R}, x \geq 0, \\ -s|x|^r, & \text{if } x \in \mathbb{R}, x \leq 0, \\ \infty, & \text{if } x = \infty \end{cases}$$

where $r, s \in \mathbb{R}$, $r, s > 0$. A *semi-multiplicative homeomorphism* of $\mathbb{R}$ is a homeomorphism f of $\mathbb{R}$ that extends to a semi-multiplicative homeomorphism of $\mathbb{S}^1$ by $f(\infty) = \infty$. We also write f_r for $f_{r,1}$, that is, $f_r(x) = x|x|^{r-1}$ is a multiplicative homeomorphism of $\mathbb{R}$. We call a homeomorphism f of $\mathbb{S}^1$ *inversely (semi)-multiplicative* if f is of the form $f(x) = 1/f_{r,s}(x)$.

The generalised Hartmann planes $\mathcal{M}(r_1, s_1; r_2, s_2)$ where $r_1, s_1, r_2, s_2 > 0$ are obtained as special Artzy–Groh planes for $f = 1/f_{r_1,s_1}$ and $g = 1/f_{r_2,s_2}$ being inversely semi-multiplicative. The circle set of such a plane thus consists of all Euclidean lines extended by the point (∞, ∞) and for $a, b, c \in \mathbb{R}$, the sets

$$\left\{ \left(x, \frac{a}{f_{r_1,s_1}(x-b)} + c \right) \middle| \ x \in \mathbb{S}^1 \right\} \quad \text{and} \quad \left\{ \left(x, \frac{a}{f_{r_2,s_2}(x-b)} + c \right) \middle| \ x \in \mathbb{S}^1 \right\}$$

for $a > 0$ and $a < 0$, respectively.

The planes $\mathcal{M}(r_1, s_1; r_2, s_2)$ generalise Hartmann's [3] construction of the first non-classical flat Minkowski planes which, in the above notation, are the planes $\mathcal{M}(r, 1; r, 1)$ for $r > 0$. (In fact, not all of Hartmann's planes from [3] are of Artzy-Groh type.) They can be characterised in terms of the configuration consisting of all points at which the Minkowski plane has Desarguesian derived planes, see [14], and in terms of their Klein-Kroll types; see [8].

Note that each map $(x, y) \mapsto (rx + b, sy + c)$, where $b, c, r, s \in \mathbb{R}$, $r, s > 0$, extends to an automorphism of $\mathcal{M}(r_1, s_1; r_2, s_2)$.

Clearly, $\mathcal{M}(1, 1; 1, 1)$ is the classical flat Minkowski plane. In fact, this is the only instance the classical flat Minkowski plane occurs among the Artzy–Groh planes; see [11].

For the classification of these planes with respect to their group dimensions see [11], Theorem 4.3.11 and Corollary 4.3.12.

With these two families of flat Minkowski planes Schenkel's classification [13] of flat Minkowski planes of group dimension at least 4 can be stated as follows.

THEOREM 2.3 *A flat Minkowski plane is isomorphic to the classical flat Minkowski plane if and only if its automorphism group has dimension at least 5.*

A flat Minkowski plane of group dimension 4 is isomorphic to either a non-classical plane $\mathcal{M}(f, id)$ where f is a semi-multiplicative homeomorphism of $\mathbb{R}$ or a non-classical generalised Hartmann plane $\mathcal{M}(r_1, s_1; r_2, s_2)$ where $r_1, s_1, r_2, s_2 > 0$, $(r_1, s_1, r_2, s_2) \neq (1, 1, 1, 1)$.

3 q-translations

Let q be a point of a Minkowski plane $\mathcal{M}$. A *q-translation* of $\mathcal{M}$ is an automorphism of $\mathcal{M}$ that is either the identity or fixes precisely the point q and induces a translation of the derived affine plane $\mathcal{A}_q$ at q. More precisely, let C be a circle passing through q. Let $B(q, C)$ denote the *touching pencil with support* q, that is, $B(q, C)$ consists of all circles that touch the circle C at the point q. In the derived affine plane at q the touching pencil represents a parallel class of lines and we can look at translations in this direction. Then a *$(q, B(q, C))$-translation* of $\mathcal{M}$ is a q-translation that fixes C (and thus each circle in $B(p, C)$) globally. A group of $(q, B(q, C))$-translations of $\mathcal{M}$ is called *$(q, B(q, C))$-transitive*, if it acts transitively on $C \setminus \{q\}$. A group of q-translations is called *q-transitive*, if it acts transitively on $P \setminus |q|$. We say that the automorphism group Γ of $\mathcal{M}$ is $(q, B(q, C))$-transitive or q-transitive if Γ contains a $(q, B(q, C))$-transitive subgroup of $(q, B(q, C))$-translations or a q-transitive subgroup of q-translations, respectively.

With respect to q-translations Klein and Kroll obtained seven types of Minkowski planes; see Klein–Kroll [1989], Theorem 4.9.

If $\mathcal{T}$ denotes the set of all points q for which the Minkowski plane is $(q, B(q, C))$-transitive for some touching pencil $B(q, C)$ with support q, then exactly one of the following statements is valid.

I. $\mathcal{T} = \emptyset$.

II. There is a point q such that $\mathcal{T} = \{q\}$ and there is exactly one touching pencil with support q such that $\mathcal{M}$ is $(q, B(q, C))$-transitive.

III. There is a point q such that $\mathcal{T} = \{q\}$ and $\mathcal{M}$ is q-transitive.

IV. $\mathcal{T}$ consists of the points on a circle.

V. $\mathcal{T}$ consists of the points on a parallel class.

VI. $\mathcal{T} = P$ and for each point q there is exactly one touching pencil $B(q, C)$ with support q such that $\mathcal{M}$ is $(q, B(q, C))$-transitive.

VII. $\mathcal{T} = P$ and $\mathcal{M}$ is q-transitive for every point q.

As a first step in showing that types V and VI cannot occur in flat Minkowski planes we have the following.

LEMMA 3.1 *If the set $\mathcal{T}$ of a flat Minkowski plane $\mathcal{M}$ contains all points on a parallel class, then $\mathcal{M}$ is the classical flat Minkowski plane and thus of type VII.*

Proof. Let $\mathcal{M}$ be a flat Minkowski plane with automorphism group Γ and suppose that $\mathcal{T}$ contains all points on the $(+)$-parallel class π_∞. Furthermore, let p and q be distinct points of π_∞ and let $\pi_0 \neq \pi_\infty$ be another $(+)$-parallel class. Since $\mathcal{M}$ is $(p, B(p, C))$-transitive for some touching pencil $B(p, C)$ with support p and $(q, B(q, \tilde{C}))$-transitive for some touching pencil $B(q, \tilde{C})$ with support q, we can find for each $(+)$-parallel class $\pi \neq \pi_\infty$ a p-translation α and a q-translation β such that $\alpha(\pi_0)$ equals $\beta(\pi_0) = \pi$. Then $\beta^{-1}\alpha$ fixes π_∞ and π_0 and $\beta^{-1}\alpha(p) = \beta^{-1}(p)$. But $\mathcal{M}$ is $(q, B(q, \tilde{C}))$-transitive so that the group of $(q, B(q, \tilde{C}))$-translations is transitive on $\pi_\infty \setminus \{q\}$. This shows that for each $p' \in \pi_\infty \setminus \{q\}$ there an automorphism γ that fixes π_∞ and π_0 and such that $\gamma(p) = p'$, that is, the stabiliser $\Gamma_{\pi_\infty, \pi_0}$ is transitive on $\pi_\infty \setminus \{q\}$. Interchanging the roles of p and q, we likewise see that $\Gamma_{\pi_\infty, \pi_0}$ is transitive on $\pi_\infty \setminus \{p\}$. Hence $\Gamma_{\pi_\infty, \pi_0}$ is 2-transitive on π_∞ and $\Gamma_{\pi_\infty, \pi_0}$ must be at least 3-dimensional by Brouwer's theorem; see [11] Theorem A2.3.8.

Since $\Gamma_{\pi_\infty, \pi_0}$ fixes the parallel class π_0, this group cannot contain any $(p, B(p, C))$-translation except the identity. This shows that Γ must be at least 4-dimensional. From Theorem 2.3 we obtain that $\mathcal{M}$ is either isomorphic to a plane of the form $\mathcal{M}(f, id)$, for a homeomorphism f such that $-f$ is multiplicative, or isomorphic to a generalised Hartmann plane $\mathcal{M}(r_1, s_1; r_2, s_2)$. However, the connected component of the full automorphism group of a non-classical generalised Hartmann plane fixes a point p (the point (∞, ∞) in the usual coordinatization) and cannot be $(q, B(q, C))$-transitive for any other point q. This shows that non-classical generalised Hartmann planes cannot occur.

The connected component of the full automorphism group of a non-classical plane of the form $\mathcal{M}(f, id)$, as above, fixes two parallel classes and thus cannot be $(q, B(q, C))$-transitive for any point q. This shows that this kind of plane cannot occur either. Therefore, the only plane possible is the classical flat Minkowski plane, which of course is of type VII. $\square$

Since in Minkowski planes of types V and VI the set $\mathcal{T}$ contains all points on a parallel class, we readily obtain the following characterization of the classical flat Minkowski plane.

COROLLARY 3.2 *There is no flat Minkowski plane of type V or VI.*
A flat Minkowski plane is classical if and only if it is of Klein–Kroll type at least V. The classical flat Minkowski plane is the only flat Minkowski plane of Klein–Kroll type VII.

For the possible Klein–Kroll types of flat Minkowski planes we therefore have the following.

PROPOSITION 3.3 *A flat Minkowski plane is of Klein–Kroll type I, II, III, IV or VII.*

For examples for each of these types see Section 6.

4 G-translations

Let G be a parallel class of a Minkowski plane $\mathcal{M}$. A *G-translation* of $\mathcal{M}$ is an automorphism of $\mathcal{M}$ that either fixes precisely the points of G or is the identity. A group of G-translations of $\mathcal{M}$ is called *G-transitive*, if it acts transitively on each parallel class H of type opposite the type of G without the point of intersection with G. We say that the automorphism group Γ of $\mathcal{M}$ is G-transitive if Γ contains a G-transitive subgroup of G-translations. With respect to G-translations Klein and Kroll obtained six types of Minkowski planes, in fact, the more general hyperbola structures, that is, geometries that satisfy all the axioms of a Minkowski plane except the axiom of touching; see Klein–Kroll [7] Theorem 3.4.
If $\mathcal{Z}$ denotes the set of all parallel classes G for which the hyperbola structure is G-transitive, then exactly one of the following statements is valid.

A. $\mathcal{Z} = \emptyset$.

B. $|\mathcal{Z}| = 1$.

C. There is a point p such that $\mathcal{Z} = \{|p|_+, |p|_-\}$.

D. $\mathcal{Z}$ consists of all (+)-parallel classes or of all (−)-parallel classes.

E. $\mathcal{Z}$ consists of all (+)-parallel classes plus one (−)-parallel class or of all (−)-parallel classes plus one (+)-parallel class.

F. $\mathcal{Z}$ consists of all (+)- and all (−)-parallel classes.

Types D, E and F all contain all (+)-parallel classes or all (–)-parallel classes. In this situation we can explicitly describe the flat Minkowski planes. These are the planes $\mathcal{M}(f, id)$; see Section 2.1.

LEMMA 4.1 *If the set $\mathcal{Z}$ of all parallel classes G for which the automorphism group of the flat Minkowski plane $\mathcal{M}$ is G-transitive contains all (–)-parallel classes, then the kernel T^+ is 3-dimensional and $\mathcal{M}$ is isomorphic to a plane $\mathcal{M}(f, id)$ (see Section 2.1) for some orientation-preserving homeomorphism f of $\mathbb{S}^1$.*

Proof. Let $\mathcal{M}$ be a flat Minkowski plane having the property that the set $\mathcal{Z}$ contains all (–)-parallel classes. Let G be a (–)-parallel class and let γ be a G-translation. Then γ fixes every point of G and thus $\gamma \in T^+$. Since $\mathcal{M}$ is G-transitive, we see that T^+ is transitive on $H \setminus \{p\}$ for every (+)-parallel class H, where $p = G \cap H$. By using a different (–)-parallel class, we obtain that T^+ is transitive on $H \setminus \{q\}$ for some other point $q \in H$. Hence T^+ is 2-transitive on H. But T^+ is also effective on H. But then T^+ must be 3-dimensional by Brouwer's Theorem, cf. [12] 96.30 or [11] A2.3.8. We conclude that $\mathcal{M}$ is isomorphic to a plane $\mathcal{M}(f, id)$ by Theorem 2.2. □

Since a 3-dimensional kernel contains a subgroup isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, see Theorem 2.2, we readily obtain the following.

COROLLARY 4.2 *A flat Minkowski plane is of Klein-Kroll type at least D if and only if the connected component of the identity in one kernel is isomorphic to* $\mathrm{PSL}_2(\mathbb{R})$.

An immediate consequence of the above Lemma 4.1 is that type E cannot occur in flat Minkowski planes.

COROLLARY 4.3 *If the set $\mathcal{Z}$ of all parallel classes G for which the automorphism group of the flat Minkowski plane $\mathcal{M}$ is G-transitive contains all (–)-parallel classes plus one (+)-parallel class, then $\mathcal{M}$ is the classical flat Minkowski plane and thus of type F.*

In particular, there is no flat Minkowski plane of type E.

Proof. From Lemma 4.1 we know that T^+ is 3-dimensional. Let π be a (+)-parallel class contained in $\mathcal{Z}$. Then each derived affine plane at a point of π is a translation plane and thus Desarguesian by [12] Proposition 32.8. Hence $\mathcal{M}$ is classical by Theorem 2.2. □

PROPOSITION 4.4 *A flat Minkowski plane is of Klein–Kroll type A, B, C, D or F.*

For examples for each of these types see Section 6.

Clearly, a type F plane is the classical flat Minkowski plane. Furthermore, by Lemma 4.1, type D comprises, up to isomorphism, precisely the non-classical flat Minkowski planes of the form $\mathcal{M}(f, id)$.

PROPOSITION 4.5 *A flat Minkowski plane is of type D if and only if it is isomorphic to a plane $\mathcal{M}(f, id)$ for some orientation-preserving homeomorphism f of the unit circle $\mathbb{S}^1$, where $f \notin \mathrm{PSL}_2(\mathbb{R})$.*

By combining both the classifications with respect to G- and q-translations Klein–Kroll [7] Theorem 4.12, obtained ten types of Minkowski planes, of which only seven types can possibly occur in flat Minkowski planes.

PROPOSITION 4.6 *A flat Minkowski plane is of Klein–Kroll type I.A, I.B, I.D, II.A, III.C, IV.A or VII.F.*

For examples for each of these combined types see Section 6.

5 (p, q)-Homotheties

Finally, a third kind of central automorphisms has been used in the classification in Klein [6]. Let p and q be two non-parallel points of a Minkowski plane $\mathcal{M}$. An automorphism γ of $\mathcal{M}$ is a *(p, q)-homothety* if γ fixes p and q and induces a homothety with centre q in the derived affine plane $\mathcal{A}_p$ at p. A group of (p, q)-homotheties is called *(p, q)-transitive* if it acts transitively on each circle through p and q minus the two points p and q. We say that the automorphism group Γ of $\mathcal{M}$ is (p, q)-transitive if Γ contains a (p, q)-transitive subgroup of (p, q)-homotheties.

With respect to (p, q)-homotheties Klein [6] obtained 23 types of Minkowski planes. Some of the types are known to occur only in finite Minkowski planes or only as the type of a proper subgroup of the full automorphism group; compare [6] Theorem 2.15, [9], [10] Corollary 2.7 and Lemma 3.5, and [4] Theorem 2. Following we list only those 14 types which can occur in infinite Minkowski planes.

If $\mathcal{H}$ denotes the set of all unordered pairs of points $\{p, q\}$ for which the flat Minkowski plane is (p, q)-transitive, then exactly one of the following statements is valid. (Note that (p, q)- and (q, p)-transitivity are equivalent.)

1. $\mathcal{H} = \emptyset$.
2. $|\mathcal{H}| = 1$.
3. There are two non-parallel points p and q such that

$$\mathcal{H} = \{\{p, q\}, \{|p|_+ \cap |q|_-, |p|_- \cap |q|_+\}\}.$$

10. There are a point p and a parallel class G with $p \notin G$ such that
$$\mathcal{H} = \{\{p, q\} \mid q \in G \setminus (G \cap |p|)\}.$$

11. There are a point p and two distinct parallel classes F and G of the same type with $p \in F$ such that $\mathcal{H} = \{\{p, s\} \mid s \in G \setminus \{q\}\} \cup \{\{q, r\} \mid r \in F \setminus \{p\}\}$ where $q = G \cap |p|$.

12. There are a parallel class G and an involution $\varphi : G \to G$ such that
$$\mathcal{H} = \{\{p, q\} \mid p \in G, q \neq \varphi(p) \text{ but parallel to } \varphi(p)\}.$$

13. There are two parallel classes F, G of the same type such that
$$\mathcal{H} = \{\{p, q\} \mid p \in F, q \in G, q \text{ not parallel to } p\}.$$

14. There is an involution $\varphi^+ : \Pi^+ \to \Pi^+$ such that
$$\mathcal{H} = \{\{p, q\} \mid p, q \in P \text{ not parallel to each other}, \varphi^+(|p|_+) = |q|_+\}$$
or there is an involution $\varphi^- : \Pi^- \to \Pi^-$ such that
$$\mathcal{H} = \{\{p, q\} \mid p, q \in P \text{ not parallel to each other}, \varphi^-(|p|_-) = |q|_-\}.$$

15. There are a circle C and a point $p \in C$ such that $\mathcal{H} = \{\{p, q\} \mid q \in C \setminus \{p\}\}$.

17. There is a point p such that $\mathcal{H} = \{\{r, s\} \mid r \in |p|_+ \setminus \{p\}, s \in |p|_- \setminus \{p\}\}$

18. There is a point p such that $\mathcal{H} = \{\{p, q\} \mid q \text{ not parallel to } p\}$.

19. There is a point p such that
$$\mathcal{H} = \{\{p, q\} \mid q \text{ not parallel to } p\} \cup \{\{r, s\} \mid r \in |p|_+ \setminus \{p\}, s \in |p|_- \setminus \{p\}\}.$$

20. There is a circle C such that $\mathcal{H} = \{\{p, q\} \mid p, q \in C, p \neq q\}$.

23. $\mathcal{H}$ consists of all pairs of non-parallel points.

In a series of Lemmas we show that types 12,13, 14, 17 and 20 cannot occur in flat Minkowski planes.

LEMMA 5.1 *If the set $\mathcal{H}$ of a flat Minkowski plane $\mathcal{M}$ contains all unordered pairs of distinct points p, q on a circle C, then $\mathcal{M}$ is the classical flat Minkowski plane and thus of type 23.*

Proof. The group Σ generated by all (p,q)-homotheties of such a Minkowski plane $\mathcal{M}$ is sharply 3-transitive on C. Hence Σ is isomorphic to $\mathrm{PGL}_2(\mathbb{R})$. We can further assume that C is the circle $y = x$ and that Σ consists of the transformations $(x,y) \mapsto (\delta(x), \delta(y))$ for $\delta \in \mathrm{PGL}_2(\mathbb{R})$.

Let D be a circle that intersects C in two points p and q. Since the group of all (p,q)-homotheties is transitive on every circle through p and q minus these two points, we see that $D \setminus \{p,q\}$ is an orbit under the stabiliser in Σ of these two points. However, this stabiliser is a subgroup of the automorphism group of the classical flat Minkowski plane so that D is the graph of a fractional linear map as in the classical flat Minkowski plane. This argument shows that every circle of $\mathcal{M}$ that intersects C in (at least) two points corresponds to a circle of the classical flat Minkowski plane. In fact, each such circle of the classical flat Minkowski plane occurs in this way. Note that therefore the negative half of $\mathcal{M}$ is the same as the negative half of the classical flat Minkowski plane.

Now let D be a circle that touches C at a point p. The subgroup of Σ generated by all (p,q)-homotheties where $q \in C \setminus \{p\}$ contains a $(p, B(p,C))$-transitive subgroup Δ of $(p, B(p,C))$-translations and $D \setminus \{p\}$ is an orbit under Δ. Furthermore, Δ corresponds to a subgroup of $\mathrm{PGL}_2(\mathbb{R})$ consisting of the identity and all fractional linear maps that fix precisely one given element of $\mathbb{S}^1$. (Such a subgroup is conjugate to $\{x \mapsto x + t \mid t \in \mathbb{R}\}$.) As before we see that D is described by a fractional linear map as in the classical flat Minkowski plane. Hence every circle of $\mathcal{M}$ that intersects C in exactly one point corresponds to a circle of the classical flat Minkowski plane. Furthermore, each such circle of the classical flat Minkowski plane occurs in this way.

We finally look at circles that do not intersect C. To this end let Φ be the subgroup of Σ consisting of the transformations $\varphi : (x,y) \mapsto (\rho_t(x), \rho_t(y))$ where $t \in \mathbb{R}$ and $\rho_t \in \mathrm{PGL}_2(\mathbb{R})$ is of the form $\rho_t : x \mapsto \dfrac{x\cos t - \sin t}{x \sin t + \cos t}$. This subgroup contains a unique automorphism φ_0 of order 3. For each point r of $\mathcal{M}$ the points r, $\varphi_0(r)$, $\varphi_0^2(r)$ are mutually non-parallel so that there is a unique circle E_r through them. This circle E_r is clearly fixed by φ_0. Furthermore, two such circles are either disjoint of equal. Since this can be done for every point r, we obtain a flock of circles $\mathcal{F} = \{E_r \mid r||_+p\}$ fixed by φ_0 and each circle fixed by φ_0 must be contained in $\mathcal{F}$. Moreover, because Φ is abelian, each circle in the orbit $\Phi(E_r)$ of E_r (for fixed r) is fixed by φ_0 too so that $\mathcal{F}$ is invariant under Φ. Clearly, $\mathcal{F}$ is homeomorphic to $|p|_+$ which is homeomorphic to $\mathbb{S}^1$. But Φ fixes C so that Φ cannot be transitive on $\mathcal{F}$. However, $\mathrm{SO}_2(\mathbb{R})$ can only act trivially on $\mathbb{R}$. Hence Φ fixes every circle in $\mathcal{F}$. On the other hand Φ must be transitive on the set of parallel classes so that the circles in $\mathcal{F}$ are orbits under Φ. As before we see that every circle in $\mathcal{F}$ is thus described by a fractional linear map as in the classical flat Minkowski plane.

Now let E be a circle of the classical flat Minkowski plane that is disjoint to C. Such a circle is the graph of a fixed-point-free fractional linear map α. Using the Jordan canonical form of a corresponding 2×2 matrix we see that α is conjugate to ρ_t from above for some $t \in (0, \pi)$, say $\alpha = \beta\rho_t\beta^{-1}$ where $\beta \in \mathrm{PSL}_2(\mathbb{R})$. The graph of $\beta^{-1}\alpha\beta$ is a circle of the classical flat Minkowski plane. Furthermore, the stabiliser of this circle contains the group Φ from above. This shows that the graph of $\beta^{-1}\alpha\beta$ is a circle of $\mathcal{M}$, namely a circle E_r in the flock $\mathcal{F}$. Hence E can be obtained from some circle E_r under the group Σ. In particular, E (in the classical flat Minkowski plane) is a circle of $\mathcal{M}$. Thus every circle of the classical flat Minkowski plane that does not intersect C corresponds to a circle of $\mathcal{M}$.

In summary we have seen that every circle of the classical flat Minkowski plane is a circle of $\mathcal{M}$. Hence $\mathcal{M}$ must be classical. □

Since in type 20 the set $\mathcal{H}$ satisfies the assumptions of Lemma 5.1 we have the following.

COROLLARY 5.2 *There is no flat Minkowski planes of type 20.*

LEMMA 5.3 *If the set $\mathcal{H}$ of a flat Minkowski plane $\mathcal{M}$ contains all unordered pairs of non-parallel points $p \in F$, $q \in G$ for two parallel classes F and G, then $\mathcal{M}$ is of type at least 19.*

Proof. Fixing a point $p \in F \setminus G$ and varying $q \in F \setminus |p|$ we see that the derived affine plane $\mathcal{A}_p$ at p is Desarguesian. We similarly obtain that each derived affine plane at a point of $G \setminus F$ is Desarguesian. If F and G are of the same type, we have two disjoint parallel classes at all whose points the derived affine plane is Desarguesian. If F and G are of different types, they intersect in one point r and we know that each derived affine plane at a point of $(F \cup G) \setminus \{r\}$ is Desarguesian. But then the derived affine plane at r must also be Desarguesian. To see this let $F = |r|_+$ and $G = |r|_-$. Fixing a point $p \in F \setminus \{r\}$ and varying $q \in G \setminus \{r\}$ we see that each translation in direction G of $\mathcal{A}_p$ comes from an automorphism of $\mathcal{M}$. Such an automorphism is in the kernel of $\mathcal{M}$ and fixes F pointwise. We therefore obtain a translation direction G of $\mathcal{A}_r$. We similarly find that each translation in direction F of $\mathcal{A}_r$ comes from an automorphism of $\mathcal{M}$. Hence $\mathcal{A}_r$ is a translation plane and thus Desarguesian.

In any case we find that the derived affine planes at points of $F \cup G$ are Desarguesian. It now follows from [14] that $\mathcal{M}$ is isomorphic to a Hartmann plane $\mathcal{M}(r, 1; r, 1)$. But this plane is of type at least 19, see the example in Section 6. □

The set $\mathcal{H}$ clearly contains a set like in the assumptions of Lemma 5.3 in case of types 13 and 17. In case of type 14 note that an involution is not the identity so that the assumptions of Lemma 5.3 are also satisfied. Hence we have the following.

COROLLARY 5.4 *There are no flat Minkowski planes of types 13, 14 or 17.*

LEMMA 5.5 *There is no flat Minkowski plane of type 12.*

Proof. Assume otherwise that there is a flat Minkowski plane $\mathcal{M}$ of type 12 and let G be the distinguished parallel class. Without loss of generality say G is a (+)-parallel class. Clearly, the automorphism group Γ of $\mathcal{M}$ must fix G.

Fixing $p \in G$ and varying $q \in \varphi(p) \setminus G$ we see, as in the proof of Lemma 5.3, that the derived affine plane $\mathcal{A}_p$ at p admits all translations in the direction of (−)-parallel classes and that these translations are automorphisms in the kernel T^- of $\mathcal{M}$. In particular, T^- is at least 1-dimenional.

If $F \neq G$ is a second (+)-parallel class, then the $(p, \varphi(q) \cap F)$-transitivity for three different points p implies that Γ is 2-transitive on G. But then the factor group Γ/T^- is effective and 2-transitive on $G \approx \mathbb{S}^1$ so that Γ/T^- is 3-dimensional by Brouwer's theroem. This shows that Γ must be at least 4-dimensional. The classification of such Minkowski planes, see Theorem 2.3, then tells us that $\mathcal{M}$ is isomorphic to a Minkowski plane $\mathcal{M}(f, id)$ where is is semi-multiplicative or a generalised Hartmann plane. However, for the planes of the first kind only types 1 and 23 can occur; see [16] 6.2.

Since a generalised Hartmann plane admits all translations $(x, y) \mapsto (x + u, y + v)$ for $u, v \in \mathbb{R}$, the parallel class G must be the only (+)-parallel class fixed under this group. But then the translation group is transitive on $G \setminus \{(\infty, \infty)\}$ so that $\mathcal{M}$ must be $((\infty, \infty), q)$-transitive for all q not parallel to (∞, ∞)—a contradiction to the assumptions made in type 12.

Since we obtain a contradiction in any case, we see that type 12 is not possible. □

From the list at the beginning of this section and from Corollaries 5.2 and 5.4 and Lemma 5.5 we find that only the following types with respect to homotheties are possible in flat Minkowski planes.

PROPOSITION 5.6 *A flat Minkowski plane is of type 1, 2, 3, 10, 11, 15, 18, 19 or 23.*

For examples for each of these types except type 15 see Section 6.

Type 23 describes the classical flat Minkowski plane and type 19 characterises the proper Hartmann planes $\mathcal{M}(r, 1; r, 1)$; see Klein-Kroll [8] Theorem 4.1. Furthermore, type 18 can also be completely characterised.

PROPOSITION 5.7 *A flat Minkowski plane is of type*

18 *if and only if it is isomorphic to an Artzy–Groh plane $\mathcal{M}^{AG}(f, g)$ where f and g are normalised and odd (that is, $f(\infty) = g(\infty) = 0$, $f(0) = g(0) = \infty$, $f(1) = g(1) = 1$ and $f(-x) = -f(x)$, $g(-x) = -g(x)$) except when $f = g$ is inversely multiplicative;*

19 if and only if it is isomorphic to a proper Hartmann plane $\mathcal{M}(r,1;r,1)$*,* $r \neq 1$*;*

23 if and only if it is classical.

Proof. The derived affine plane at the distinguished point p in a flat Minkowski plane $\mathcal{M}$ of type 18 is Desarguesian and $\mathcal{M}$ admits the transformations

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto (rx+b, ry+c),$$

where $b, c, r \in \mathbb{R}$, $r \neq 0$, as automorphisms. By [1] Theorem 4.8 a flat Minkowski plane that admits the above transformations as automorphisms for $r > 0$ is isomorphic to an Artzy–Groh plane. But $\sigma : (x,y) \mapsto (-x,-y)$ must also be an automorphism. In particular, in the notation of Section 2.2, $\sigma(C_{1,0,0}) = C_{a,0,0}$ for some $a \neq 0$. In fact, $a > 0$ since $C_{a,0,0}$ must pass through $(-1,-1) = \sigma(1,1)$. Let φ_a be the automorphism given by $\varphi_a(x,y) = (ax, ay)$. Then σ and φ_a commute and

$$C_{1,0,0} = \sigma^2(C_{1,0,0}) = \sigma(C_{a,0,0}) = \sigma\varphi_a(C_{1,0,0}) = \varphi_a\sigma(C_{1,0,0}) = \varphi_a(C_{a,0,0}) = C_{a^2,0,0}.$$

This implies $a^2 = 1$ and thus $a = 1$. Hence σ fixes $C_{1,0,0}$, that is $-f(-x) = f(x)$ and f is odd. The same argument applied to $C_{-1,0,0}$ shows that g must be odd too. By [8] Theorem 4.1 those normalised functions $f = g$ that are inversely multiplicative, that is, Hartmann planes $\mathcal{M}(r,1;r,1)$, yield Minkowski planes of type at least 19. □

Combining all three classifications Klein [5] Theorem 2.16, obtained 32 types. Of those only 14 types can possibly occur in flat Minkowski planes.

THEOREM 5.8 *A flat Minkowski plane is of Klein–Kroll type*

I. A.1, A.2, A.3, B.1, B.10, B.11, D.1,

II. A.1, A.15,

III. C.1, C.18, C.19,

IV. A.1 or

VII. F.23.

For examples for these combined types except type II.A.15 see the following section. Using combined types Corollaries 3.2 and 4.3, Lemma 5.1 and Proposition 5.7 can now be restated as follows.

PROPOSITION 5.9 *A flat Minkowski plane of Klein-Kroll type*

- *VII.F.23 is isomorphic to the classical flat Minkowski plane.*
- *III.C.19 is isomorphic to a proper Hartmann planes* $\mathcal{M}(r,1;r,1)$*,* $r \neq 1$*;*
- *III.C.18 is isomorphic to an Artzy–Groh plane* $\mathcal{M}^{AG}(f,g)$ *where* f *and* g *are normalised and odd except when* $f=g$ *is inversely multiplicative;*
- *I.D.1 is isomorphic to a plane* $\mathcal{M}(f,id)$ *where* f *is an orientation preserving homeomorphism of* $\mathbb{S}^1$ *not in* $\mathrm{PSL}_2(\mathbb{R})$*.*

6 Examples

In this section we provide examples for all of the possible Klein-Kroll types of flat Minkowski planes as given in Theorem 5.8 except for type II.A.15 thus establishing that a flat Minkowski plane is precisely of one of the 14 types. In most cases we just state the respective transitive groups of central automorphisms without explicitly verifying that no further transitive groups of central automorphisms exist. In many cases this will follow from the group dimensions and/or kernel dimensions of the Minkowski planes involved. Note however that some of the flat Minkowski planes admit 1-dimensional subgroups of central automorphisms without being linearly transitive (involutory central automorphisms, which interchange corresponding connected components, are not admissible in these cases). Of course, there normally are many different flat Minkowski planes of a given Klein–Kroll type, but we only give one kind of model for each type.

I.A.1: A plane of group dimension 0. Clearly, such a plane must be of Klein–Kroll type I.A.1. For example, the plane $\mathcal{M}(f,f)$, where $f(x)=\sinh x$ for $x \in \mathbb{R}$ and $f(\infty)=\infty$ is of group dimension 0; see [15], Theorem 3.9.

I.A.2: A Minkowski plane obtained from an Artzy–Groh plane $\mathcal{M}^{AG}(f,f)$ where f is not inversely semi-multiplicative by swapping the negative half with the plane $\gamma(\mathcal{M}^{AG}(f,f))$ isomorphic to $\mathcal{M}^{AG}(f,f)$ where γ is the transformation $(x,y)\mapsto(1/x,1/y)$. See [11], Theorem 4.3.11, for the group dimension classification of Artzy–Groh planes. The distinguished points are (∞,∞) and $(0,0)$. Note that $\mathcal{M}^{AG}(f,f)$ is $((\infty,\infty),p)$-transitive for each $p \in \mathbb{R}^2$ and thus $\gamma(\mathcal{M}^{AG}(f,f))$ is $((0,0),q)$-transitive for each q not parallel to $(0,0)$. Hence, for each $r \in \mathbb{R}$, $r \neq 0$, the transformation $(x,y)\mapsto(rx,ry)$ is a $\{(\infty,\infty),(0,0)\}$-homothety.

I.A.3: The plane $\mathcal{M}(f_3,f_3)$; see [16] 6.2. This plane has group dimension 2. The distinguished points p and q are the points (∞,∞) and $(0,0)$. For each $r \in \mathbb{R}$, $r \neq 0$, the transformation $(x,y)\mapsto(rx,ry)$ is a $\{(\infty,\infty),(0,0)\}$-homothety. Likewise, the map $(x,y)\mapsto(rx,y/r)$ is a $\{(\infty,0),(0,\infty)\}$-homothety.

I.B.1: A Minkowski plane obtained from an Artzy–Groh plane $\mathcal{M}^{AG}(g,g)$ by swapping the negative half with a plane $\mathcal{M}(f,id)$ where $f \notin \mathrm{PGL}_2(\mathbb{R})$. For example $f = f_3$ and $g =$ yields a flat Minkowski plane of Klein–Kroll type I.B.1. The distinguished parallel class is $G = \mathbb{S}^1 \times \{\infty\}$. For each $c \in \mathbb{R}$ the transformation $(x,y) \mapsto (x, y+c)$, where $\infty + c = \infty$, is a G-translation.

I.B.10: A Minkowski plane obtained from an Artzy–Groh plane $\mathcal{M}^{AG}(f,f)$ where f is not inversely semi-multiplicative by swapping the negative half with the isomorphic model $\gamma(\mathcal{M}(k,1;k,1))$, $k > 1$, of the proper Hartmann plane $\mathcal{M}(k,1;k,1)$ where γ is the transformation $(x,y) \mapsto (x|x|^{-k-1}, y)$. The distinguished point is (∞,∞) and the distinguished parallel classes are $G = \{0\} \times \mathbb{S}^1$ and $H = \mathbb{S}^1 \times \{\infty\}$. Note that $\mathcal{M}^{AG}(f,f)$ is $((\infty,\infty),p)$-transitive for each $p \in \mathbb{R}^2$ and that $\gamma(\mathcal{M}(r,1;r,1))$ is $((0,\infty),q)$-transitive for each $q \in (\mathbb{S}^1 \setminus \{0\}) \times \mathbb{R})$ and $((0,a),(b,\infty))$-transitive for all $a \in \mathbb{R}$, $b \in \mathbb{S}^1 \setminus \{0\}$. Hence, the combined Minkowski plane must be $((\infty,\infty),(0,y))$-transitive for each $y \in \mathbb{R}$. Furthermore, the transformation $(x,y) \mapsto (x,y+c)$ for $c \in \mathbb{R}$ is a H-translation.

I.B.11: A Minkowski plane $\mathcal{M}_k$, $k > 1$, obtained from a proper Hartmann plane $\mathcal{M}(k,1;k,1)$ by swapping the negative half with the isomorphic model of the proper Hartmann plane $\mathcal{M}(1/k,1;1/k,1)$ under is the transformation $\gamma : (x,y) \mapsto (x|x|^{-\frac{1}{k}-1}, y)$. The distinguished points are (∞,∞) and $(0,\infty)$. Note that the plane $\mathcal{M}(k,1;k,1)$ is $((\infty,\infty),p)$-transitive for each $p \in \mathbb{R}^2$ and $((\infty,a),(b,\infty))$-transitive for all $a,b \in \mathbb{R}$. Similarly, the second plane $\gamma(\mathcal{M}(1/k,1;1/k,1))$ is $((0,\infty),q)$-transitive for each q not parallel to $(0,\infty)$ and $((0,a),(b,\infty))$-transitive for all $a \in \mathbb{R}$, $b \in \mathbb{S}^1 \setminus \{0\}$. Hence, $\mathcal{M}_k$ must be $((\infty,\infty),(0,y))$-transitive and $((0,\infty),(\infty,y))$-transitive for each $y \in \mathbb{R}$.
Explicitly the circles of $\mathcal{M}_k$ are of the form

- $\{(x, mx+t) \mid x \in \mathbb{R}\} \cup \{(\infty,\infty)\}$ for $m,t \in \mathbb{R}$, $m \neq 0$;
- $\left\{(x, \frac{a}{f_k(x)} + c) \mid x \in \mathbb{R} \setminus \{0\}\right\} \cup \{(\infty,c),(0,\infty)\}$ for $a,c \in \mathbb{R}$, $a \neq 0$;
- $\left\{(x, \frac{a}{f_k(x-b)} + c) \mid x \in \mathbb{R} \setminus \{b\}\right\} \cup \{(\infty,c),(b,\infty)\}$ for $a,b,c \in \mathbb{R}$, $a > 0$, $b \neq 0$;
- $\left\{(x, a/f_k(\frac{1}{f_k^{-1}(x)} - b) + c) \mid x \in \mathbb{R} \setminus \{1/f_k(b)\}\right\} \cup \left\{(\infty, c - \frac{a}{f_k(b)}), (1/f_k(b),\infty)\right\}$ for $a,b,c \in \mathbb{R}$, $a < 0$, $b \neq 0$;

where $f_k(x) = x|x|^{k-1} = f_{k,1}(x)$. From this description one clearly sees that each transformation $\varphi_{r,s,t} : (x,y) \mapsto (rx, sy+t)$ for $r,s,t \in \mathbb{R}$, $rs > 0$, is an automorphism of $\mathcal{M}_k$. More precisely, $\{\varphi_{1,1,t} \mid t \in \mathbb{R}\}$ is a linearly transitive group of G-translations, $\{\varphi_{r,r,(1-r)c} \mid r \neq 0\}$ is a linearly transitive group of $((\infty,\infty),(0,c))$-homotheties and $\{\varphi_{1/r,f_k(r),(1-f_k(r))c} \mid r \neq 0\}$ is a linearly transitive group of $((0,\infty),(\infty,c))$-homotheties.

I.D.1: The planes $\mathcal{M}(f, id)$, where $f \notin \mathrm{PGL}_2(\mathbb{R})$. For example $f = f_3$ yields a Minkowski plane of Klein–Kroll type I.D.1. Such a plane has group dimension 3 or 4 depending on whether or not f is projectively equivalent to a semi-multiplicative homeomorphism. Here all transformations $(x, y) \mapsto (x, \delta(y))$, where $\delta \in \mathrm{PSL}_2(\mathbb{R})$, are automorphisms of $\mathcal{M}(f, id)$. For those permutations δ that fix exactly one point of $\mathbb{S}^1$, say $\delta(y_0) = y_0$, (that is, δ is conjugate to $x \mapsto x + t$ for some $t \in \mathbb{R}$) we obtain $\mathbb{S}^1 \times \{y_0\}$-translations.

II.A.1: A plane obtained from a generalised Hartmann plane $\mathcal{M}(1, 1; r, s)$, $r, s > 0$, $r \neq 1$, by swapping the negative half with a Minkowski plane $\mathcal{M}(k)$ where $k > 1$ from [17], see type IV.A.1 below for a brief description of the planes $\mathcal{M}(k)$. The distinguished touching pencil is the pencil of all circles that touch the circle $C = \{(x, x) \mid x \in \mathbb{S}^1\}$ at the point (∞, ∞).

III.C.1: The generalised Hartmann planes $\mathcal{M}(r_1, s_1; r_2, s_2)$, $r_1, r_2, s_1, s_2 > 0$, $r_1 \neq r_2$. This plane has group dimension 4. The distinguished point is the point $p = (\infty, \infty)$. Each translation of the derived affine plane at p extends to a central automorphism of the Minkowski plane. Although the automorphism group of the plane $\mathcal{M}(r_1, s_1; r_2, s_2)$ contains all transformations $(x, y) \mapsto (rx + b, sy + c)$ for $b, c, r, s \in \mathbb{R}$, $r, s > 0$, the plane does not permit the above transformations with $r = s < 0$ as automorphisms.

III.C.18: The Artzy–Groh planes $\mathcal{M}^{AG}(f, f)$ where f is odd but not inversely semi-multiplicative. This plane has group dimension 3. The distinguished point is the point $p = (\infty, \infty)$. Each translation and each homothety of the (Desarguesian) derived affine plane at p extends to a central automorphism of the Minkowski plane.

III.C.19: The Hartmann planes $\mathcal{M}(d, 1; d, 1)$, $d > 0$, $d \neq 1$. This plane has group dimension 4. The distinguished point is the point $p = (\infty, \infty)$. Each translation and each homothety of the (Desarguesian) derived affine plane at p extends to a central automorphism of the Minkowski plane. Furthermore, the transformation $(x, y) \mapsto (rx, r|r|^{-d-1}y)$ for $r \in \mathbb{R}$, $r \neq 0$, is a $\{(\infty, 0), (0, \infty)\}$-homothety. Using translations one obtains $\{(\infty, c), (b, \infty)\}$-homotheties for any $b, c \in \mathbb{R}$.

IV.A.1: A Minkowski plane $\mathcal{M}(k)$ where $k > 1$ from [17]. This non-classical flat Minkowski plane is obtained from the classical flat Minkowski plane by replacing the negative half by the images of the the curve

$$C_k = \{(x, -x|x|^k) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\}$$

under the group

$$\Sigma = \{(x, y) \mapsto (\delta(x), \delta(y)) \mid \delta \in \mathrm{PSL}_2(\mathbb{R})\}.$$

The circles of $\mathcal{M}(k)$, $k > 1$, therefore are the graphs of elements in $\mathrm{PSL}_2(\mathbb{R})$ and of $\delta f_{k,1}\delta^{-1}$ for $\delta \in \mathrm{PSL}_2(\mathbb{R})$. Each of these planes has group dimension 3 and admits Σ as a group of automorphisms; see [17], Theorem 3.5.

The distinguished circle is the circle $C = \{(x,x) \mid x \in \mathbb{S}^1\}$. All transformations $(x,y) \mapsto (\delta(x), \delta(y))$, where $\delta \in \mathrm{PSL}_2(\mathbb{R})$, are automorphisms of $\mathcal{M}(k)$ and each fixes the circle C. For those permutations δ that fix exactly one point of $\mathbb{S}^1$, say $\delta(x_0) = x_0$, we obtain $((x_0,x_0), B((x_0,x_0),C)$-translations. Note that $\mathcal{M}(k)$ does not admit any non-trivial homotheties. For example, the map $(x,y) \mapsto (sx, sy)$ for $s \neq 0, 1$ is an automorphism that fixes (∞,∞) and $(0,0)$, but it is not a $((\infty,\infty),(0,0))$-homothety because it only fixes those circles through these points that are in the positive half.

VII.F.23: The classical flat Minkowski plane. Here all admissible subgroups of central automorphisms are linearly transitive.

References

[1] R. Artzy and H. Groh. Laguerre and Minkowski planes produced by dilatations. J. Geom. **26** (1986) 1–20.

[2] W. Benz. Vorlesungen über Geometrie der Algebren, Springer, Berlin, 1973.

[3] E. Hartmann. Beispiele nicht einbettbarer reeller Minkowski-Ebenen. Geom. Dedicata **10** (1981) 155–159.

[4] J. Jakóbowski, H.–J. Kroll and A. Matrás. Minkowski planes admitting automorphism groups of small type. J. Geom. **71** (2001) 78–84.

[5] M. Klein. A classification of Minkowski planes, PhD Thesis, University of Haifa, 1985.

[6] M. Klein. A classification of Minkowski planes by transitive groups of homotheties. J. Geom. **43** (1992) 116–128.

[7] M. Klein, and H.–J. Kroll. A classification of Minkowski planes. J. Geom. **36** (1989) 99–108.

[8] M. Klein, and H.–J. Kroll. On Minkowski planes with transitive groups of homotheties. Abh. Math. Sem. Univ. Hamburg **64** (1994) 303–313.

[9] H.–J. Kroll. Eine Kennzeichnung der miquelschen Minkowski-Ebenen durch Transitivitätseigenschaften. Geom. Dedicata **58** (1995) 75–77.

[10] H.–J. Kroll and A. Matrás. Minkowski planes with Miquelian pairs. Beitr. Algebra Geom. **38** (1997) 99–109.

[11] B. Polster and G.F. Steinke. Geometries on Surfaces, Cambridge University Press, Encyclopedia of Mathematics and its Applications vol 84, 2001.

[12] H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, M. Stroppel. Compact Projective Planes, de Gruyter, Berlin, 1995.

[13] A. Schenkel. Topologische Minkowski-Ebenen, Dissertation, Erlangen-Nürnberg, 1980.

[14] G.F. Steinke. 2-dimensional Minkowski planes and Desarguesian derived affine planes. Abh. Math. Sem. Univ. Hamburg **60** (1990) 61–69.

[15] G.F. Steinke. A family of 2-dimensional Minkowski planes with small automorphism groups. Result. Math. **26** (1994) 131–142.

[16] G.F. Steinke. A classification of Minkowski planes over half-ordered fields. J. Geom. **69** (2000) 192–214.

[17] G.F. Steinke. A family of flat Minkowski planes admitting 3-dimensional simple groups of automorphisms. Adv. Geom. **4** (2004) 319–340.