Sisters of some 4-dimensional elation Laguerre planes of group dimension 10 *

Günter F. Steinke

Abstract

We give an explicit description of some sisters of 4-dimensional elation Laguerre planes of group dimension 10 with respect to points of the Laguerre planes. In this way we obtain the first concrete examples of 4-dimensional Laguerre planes of group dimensions 8 and 7 and of 4-dimensional Laguerre planes that are not elation Laguerre planes.

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1 Introduction

In [16] the first examples of non-classical 4-dimensional Laguerre planes were constructed. They are elation Laguerre planes and were obtained by pasting together two halves of the classical complex Laguerre plane along a 3-dimensional separating set in the point set and are therefore called *semi-classical* planes. Two further single 4-dimensional elation Laguerre planes were found in [20] and [21] and these examples together comprise all 4-dimensional elation Laguerre planes of group dimension 10, that is, having a 10-dimensional automorphism group; see Example 2 for a description of these planes. In fact, all 4-dimensional Laguerre planes known so far are elation Laguerre planes and have group dimension at least 10.

The aim of this paper is to use the close relationship between 4-dimensional Laguerre planes and generalized quadrangles with topological parameter 2 and form sisters of some of the 4-dimensional elation Laguerre planes of group dimension 10 in the sense of A.E. Schroth via associated generalized quadrangles; compare [14], Chapter 6. This process will yield the first concrete examples of 4-dimensional Laguerre planes that are not elation Laguerre planes and of planes of group dimension less than 10. In fact, the resulting Laguerre planes have group dimensions 8 and 7 and one derived affine plane is a translation plane whose translations are all induced by automorphisms of the Laguerre plane. In

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analogy to the 2-dimensional case we call these planes Laguerre planes of translation type.
The plane of group dimension 8 obtained in this paper contains as a circle a copy of Knarr's surface (see [3] or [10], 74.24) which generates the unique 4-dimensional shift plane of group dimension 7, compare [10], Theorem 74.27. Thus this shift plane occurs as a derived affine plane. The circles of the Laguerre planes of group dimension 7 presented here are made up of two halves of classical circles but the set where these halves are pasted together varies with the circle, and this is completely different from the process used to obtain the semi-classical 4-dimensional Laguerre planes. These planes contain many copies of Betten's complex skew-parabola planes, see [2], as derived affine planes. The explicit description of the circles in these models should help in the further investigation and construction of 4-dimensional Laguerre planes.
In the following section we review for easy reference the basic theory of 4-dimensional elation Laguerre planes. Section 3 gives an outline on how to form a sister of a Laguerre plane with respect to a point of the Laguerre plane and what can be said about the dimension of its automorphism group. In the last two sections we then form a sister of the single 4-dimensional elation Laguerre plane that has a solvable 10-dimensional automorphism group and sisters of the semi-classical Laguerre planes.

2 Four-dimensional elation Laguerre planes

A *4-dimensional Laguerre plane* is a Laguerre plane $\mathcal{L} = (P, \mathcal{C}, ||)$ whose point set P and circle set $\mathcal{C}$ carry Hausdorff topologies such that P is 4-dimensional locally compact and such that the geometric operations of joining three points by a circle, of intersecting two different circles, parallel projection and touching are continuous with respect to the induced topologies on their respective domains of definition. In this case the circle set $\mathcal{C}$ is homeomorphic to $\mathbb{R}^6$. Elements of $\mathcal{C}$ are considered as subsets of P and are homeomorphic to the 2-sphere $\mathbb{S}_2$. For more information about topological Laguerre planes we refer to [5], [6] and [18]. The *classical (4-dimensional) complex Laguerre plane* is the miquelian Laguerre plane over $\mathbb{C}$ (with suitable topologies) and can be obtained as the geometry of non-trivial plane sections of an elliptic cone in 3-dimensional complex projective space with its vertex removed.
The collection of all continuous automorphisms of a 4-dimensional Laguerre plane $\mathcal{L}$ is a Lie group with respect to the compact-open topology, the automorphism group Γ of $\mathcal{L}$, see [4], Satz 3.9, or [15]. The *kernel* of $\mathcal{L}$ consists of all automorphisms that fix each parallel class. This collection of automorphisms is a closed normal subgroup T of Γ. The maximum dimensions of the automorphism group and kernel of a 4-dimensional Laguerre plane are 14 and 8, respectively. These dimensions are attained in the classical complex Laguerre plane.
Every 4-dimensional Laguerre plane contains one further distinguished closed normal subgroup, the *elation group* Δ, which is the collection of all automorphisms in the kernel T that fix no circle, plus the identity. We obtain an *elation Laguerre plane* if Δ acts

sharply transitively on the set of circles, see [17]. The automorphisms in this group induce elations in the associated Lie geometry and one obtains an elation generalized quadrangle; cf. [14] for generalized quadrangles and their relation to Laguerre planes and the other types of circle planes. 4-dimensional elation Laguerre planes can be characterized in terms of transitivity properties or the dimension of T and Δ; see [17] and [19], Theorem 2.7, compare also [9]. In particular, in a 4-dimensional elation Laguerre plane Δ is isomorphic to $\mathbb{R}^6$ and sharply transitive on the set of circles, and T is at least 7-dimensional.

The stabilizer Δ_p of a point p induces elations of the *derived projective plane* $\mathcal{P}_p$ *at* p, that is, the projective completion of the *derived affine plane* $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ *at* p whose point set $A_p \approx \mathbb{R}^4$ consists of all points of $\mathcal{L}$ that are not parallel to p and whose line set $\mathcal{L}_p$ consists of all restrictions to A_p of circles of $\mathcal{L}$ passing through p and of all parallel classes not passing through p. The line at infinity is the common axis to all these elations. In fact, each derived projective plane is a topological locally compact connected 4-dimensional projective plane and, in case of an elation Laguerre plane, even a dual translation plane with centre ω, the point at infinity of the lines that come from parallel classes of the Laguerre plane.

The axioms of a 4-dimensional Laguerre plane further imply that circles not passing through the distinguished point p induce closed ovals in $\mathcal{P}_p$ by removing the point parallel to p and adding in $\mathcal{P}_p$ the point ω; compare [10], Proposition 55.18 and Theorem 55.11. The line at infinity of $\mathcal{P}_p$ (relative to $\mathcal{A}_p$) is a tangent to each of these ovals. We call the induced geometry obtained by removing from $\mathcal{L}$ the parallel class of p an *affine part* of $\mathcal{L}$; it consists of the point set of $\mathcal{A}_p$, its lines and the ovals just mentioned minus the point ω. The removed parallel class is then referred to as the parallel class at infinity.

In [17] a matrix representation of 4-dimensional elation Laguerre planes was given which also forms the basis for our present investigations; see also [9].

THEOREM 1 *Every 4-dimensional elation Laguerre plane can be represented by a matrix-valued map D, called an admissible parametrization, as follows. For each $z \in \mathbb{S}_2 \approx \mathbb{R}^2 \cup \{\infty\}$ let*

$$D(z) = \begin{pmatrix} A(z) \\ B(z) \\ C(z) \end{pmatrix}$$

be a 6×2 matrix with 2×2 matrices A, B, C such that $A(\infty) = I_2$ and $B(\infty) = C(\infty) = 0$ where 0 and I_2 denote the 2×2 zero and identity matrix, respectively, and such that for all $z \in \mathbb{R}^2$ the second row of $B(z)$ equals z and $C(z) = I_2$. One can further assume that $A(0) = B(0) = 0$.

Circles of the elation Laguerre plane $\mathcal{L}$ are of the form

$$C_c = \{(z, c \cdot D(z)) \mid z \in \mathbb{S}_2\}$$

where $c \in \mathbb{R}^6$. The elation group Δ of $\mathcal{L}$ is given by all maps

$$(z, w) \mapsto (z, w + c \cdot D(z))$$

for $c \in \mathbb{R}^6$; the connected component of the kernel containing the identity consists of all maps

$$(z, w) \mapsto (z, rw + c \cdot D(z))$$

for $c \in \mathbb{R}^6$, $r \in \mathbb{R}$, $r > 0$, if $\mathcal{L}$ is non-classical.

Note that a circle C_c passes through the point $(\infty, (c_1, c_2))$ at infinity where c_1 and c_2 denote the first and second component of the vector $c \in \mathbb{R}^6$. Hence, circles through the point $(\infty, 0)$ are of the form

$$\{(z, (c_3, c_4)B(z) + (c_5, c_6)) \mid z \in \mathbb{S}_2\}$$

for $c_3, c_4, c_5, c_6 \in \mathbb{R}$. Since this derived affine plane is a dual translation plane, $\{B(z) \mid z \in \mathbb{R}^2\}$ is a spread set. Further to the normalisations made in Theorem 1 one can achieve that

$$\left\{\{(z, (c_1, c_2)A(z)) \mid z \in \mathbb{S}_2\} \mid c_1, c_2 \in \mathbb{R}\right\}$$

represents the bundle of circles that touch C_0 at $(0, 0)$.
Points on the affine part of $\mathcal{L}$ will be denoted by (z, w) with $z, w \in \mathbb{R}^2$ or by (x, y, u, v) for $x, y, u, v \in \mathbb{R}$, where $z = (x, y)$ and $w = (u, v)$, whichever is more convenient at the time. Points at infinity form the parallel class $\Pi_\infty = \{\infty\} \times \mathbb{R}^2$. Likewise, we use $c = (c_1, c_2, c_3, c_4, c_5, c_6) \in \mathbb{R}^6$ and $c = (c^1, c^2, c^3)$ with $c^1 = (c_1, c_2), c^2 = (c_3, c_4), c^3 = (c_5, c_6) \in \mathbb{R}^2$ simultaneously for a coefficient vector c of a circle C_c so that

$$\begin{aligned} C_c &= \{(z, c \cdot D(z)) \mid z \in \mathbb{S}_2\} \\ &= \{(z, c^1 A(z) + c^2 B(z) + c^3) \mid z \in \mathbb{R}^2\} \cup \{(\infty, c^1)\}. \end{aligned}$$

Finally, note that [17], Proposition 5.10, relates touching of circles to certain differentiability conditions. In short, if $B(z)$ is differentiable at a point $z_0 \in \mathbb{R}^2$, then so is $A(z)$, and C_c touching $C_{c'}$ at (z_0, w_0) implies that the derivatives of $c \cdot D(z)$ and $c' \cdot D(z)$ at z_0 are equal.
If $D_x(z)$ and $D_y(z)$ (and likewise for the matrix-valued functions A and B) denote the partial derivatives of $D(z)$ with respect to the first and second coordinates x and y, one often finds that $D_x(z) = D_y(z)B_x(z)$ or some similar dependence between $D_x(z)$ and $D_y(z)$. For example, the semi-classical 4-dimensional elation Laguerre planes and the 4-dimensional elation Laguerre planes of group dimension 10 and solvable automorphism group, see Example 2, have this property. In this case two circles C_c and C_d touch each other at a point (z_0, w_0) for $z_0 \in \mathbb{R}^2$ not on the separating set of the plane if and only if the coordinate vectors $c, d \in \mathbb{R}^6$ satisfy $c \cdot (D_y(z_0), D(z_0)) = d \cdot (D_y(z_0), D(z_0))$

EXAMPLE 2 The *semi-classical Laguerre planes* from [16] can be described in the above fashion as follows. For $0 < q \leq 1$ and $x, y \in \mathbb{R}$ let

$$A(x,y) = \begin{cases} \begin{pmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{pmatrix}, & \text{for } y \geq 0 \\ \begin{pmatrix} qy^2 - x^2 & -2xy \\ 2qxy & qy^2 - x^2 \end{pmatrix}, & \text{for } y \leq 0 \end{cases}$$

$$B(x,y) = \begin{cases} \begin{pmatrix} y & -x \\ x & y \end{pmatrix}, & \text{for } y \geq 0 \\ \begin{pmatrix} qy & -x \\ x & y \end{pmatrix}, & \text{for } y \leq 0 \end{cases}$$

The induced topology and geometry on each of the open subsets $\{(x,y,u,v) \in \mathbb{R}^4, y > 0\}$ and $\{(x,y,u,v) \in \mathbb{R}^4, y < 0\}$ is isomorphic to the topology and geometry of the classical complex Laguerre plane on a corresponding set. Every semi-classical Laguerre plane is isomorphic to an elation Laguerre plane as described above for exactly one q, $0 < q \leq 1$. One obtains the classical complex Laguerre plane for $q = 1$. A 4-dimensional elation Laguerre plane of group dimension 10 is semi-classical if and only if it admits a (centre-free) simple Lie group of automorphisms.
Likewise the $\mathrm{SL}_2(\mathbb{R})$-elation Laguerre plane from [20] is described by

$$A(x,y) = \begin{cases} \begin{pmatrix} y^2 + 3x^2y + x^4 & xy(2y + x^2) \\ x(2y + x^2) & y(y + x^2) \end{pmatrix}, & \text{for } 4y + x^2 \leq 0 \\ \begin{pmatrix} 5(5y + \frac{3}{2}x^2)^2 & -2x(5y + \frac{3}{2}x^2)^2 \\ x(10y + 3x^2) & (5y + \frac{3}{2}x^2)(y - \frac{1}{2}x^2) \end{pmatrix}, & \text{for } 4y + x^2 \geq 0 \end{cases}$$

$$B(x,y) = \begin{cases} \begin{pmatrix} y + x^2 & xy \\ x & y \end{pmatrix}, & \text{for } 4y + x^2 \leq 0 \\ \begin{pmatrix} 5y + 2x^2 & -x(y + \frac{1}{2}x^2) \\ x & y \end{pmatrix}, & \text{for } 4y + x^2 \geq 0 \end{cases}$$

A 4-dimensional elation Laguerre plane of group dimension 10 is isomorphic to this plane if and only if it admits $\mathrm{SL}_2(\mathbb{R})$ as a group of automorphisms.
Finally, the single Laguerre plane from [21] can be described in the above fashion using

$$A(x,y) = \begin{pmatrix} \frac{1}{2}y^2 + x^2y + \frac{5}{12}x^4 & -\frac{1}{3}x^3y - \frac{1}{5}x^5 \\ xy + \frac{2}{3}x^3 & \frac{1}{2}y^2 - \frac{1}{4}x^4 \end{pmatrix},$$

$$B(x,y) = \begin{pmatrix} y + x^2 & -\frac{1}{3}x^3 \\ x & y \end{pmatrix}$$

A 4-dimensional elation Laguerre plane of group dimension 10 is isomorphic to this plane if and only if its automorphism group is solvable or fixes a parallel class.

3 Sisters of 4-dimensional elation Laguerre planes with respect to points of the Laguerre plane

It is well known that 4-dimensional Laguerre planes correspond to certain generalized quadrangles, see [11, 12, 14]. More precisely, the Lie geometry associated with a Laguerre plane is an antiregular compact generalized quadrangle with topological parameter 2 (so that all lines and line pencils are homeomorphic to the 2-dimensional sphere $\mathbb{S}_2$). Up to duality every compact generalized quadrangle with topological parameter 2 is the Lie geometry of a 4-dimensional Laguerre plane; see [14], Corollary 2.6 and Chapter 3. Recall that the *Lie geometry* of a Laguerre plane $\mathcal{L}$ has points the points of $\mathcal{L}$ plus the circles of $\mathcal{L}$ plus one additional point at infinity, denoted by $\overline{\infty}$. (The bar is added in order to distinguish from the other uses of the symbol ∞.) The lines of the Lie geometry are the extended parallel classes, that is, the parallel classes to which the point $\overline{\infty}$ is added, and the extended tangent pencils, that is, the collections of all circles that touch a given circle at a given point p together with the point p, called the support of the tangent pencil. Incidence is the natural one. So 'collinear' in the Lie geometry corresponds to 'on the same parallel class or incident or touching' in the Laguerre plane. Conversely, for every point p of an antiregular generalized quadrangle $\mathcal{Q}$ one obtains a Laguerre plane, called the *derivation at* p, whose points are the points of $\mathcal{Q}$ that are collinear with p except p itself and whose circles are of the form $p^\perp \cap q^\perp$ for points q not collinear with p, where $x^\perp$ denotes the set of all points collinear with the point x. See also [7], Theorem 3.1, where it is shown that it suffices to have a strongly antiregular point of the generalized quadrangle in order to obtain a Laguerre plane as derivation at that point.

Starting with a 4-dimensional Laguerre plane $\mathcal{L}$ one obtains an antiregular compact generalized quadrangle $\mathcal{Q}$ with topological parameter 2. One can then derive at any point p of $\mathcal{Q}$ to obtain another 4-dimensional Laguerre plane $\mathcal{L}'_p$. We call $\mathcal{L}'_p$ a *sister* of $\mathcal{L}$; see [14], Chapter 6, or [13]. The process of going from $\mathcal{L}$ to its sister $\mathcal{L}'_p$ can be easily and completely described within $\mathcal{L}$ without explicitly using the associated generalized quadrangle in case one derives the generalized quadrangle at a point that comes from a point of $\mathcal{L}$. So let p be a point of $\mathcal{L}$ and let $\mathcal{L}'_p$ be the sister with respect to p obtained in the fashion above. Then the points of $\mathcal{L}'_p$ are the circles of $\mathcal{L}$ that pass through p, the points of $\mathcal{L}$ on the parallel class $|p|$ of p but not p itself, and the extra point $\overline{\infty}$. The parallel classes of $\mathcal{L}'_p$ are obtained from the parallel class $|p|$ and the tangent pencils with support p. The circles of $\mathcal{L}'_p$ correspond to the points of $\mathcal{L}$ not on $|p|$ (more precisely, such a point q represents the collection of all circles of $\mathcal{L}$ through p and q) and to the circles of $\mathcal{L}$ not passing through p (more precisely, such a circle C represents the collection of all circles of $\mathcal{L}$ through p that touch C). Thus the affine part of $\mathcal{L}'_p$ with respect to the parallel class containing $\overline{\infty}$ is made up of the non-vertical lines of the derived affine plane $\mathcal{A}_p$ of $\mathcal{L}$ at p, and points of $\mathcal{A}_p$ represent circles of $\mathcal{L}'_p$ through $\overline{\infty}$. Hence the derived projective plane $\mathcal{P}'_{\overline{\infty}}$ of $\mathcal{L}'_p$ at $\overline{\infty}$ is the dual of $\mathcal{P}_p$, the derived projective plane of $\mathcal{L}$ at p. A circle of $\mathcal{L}$ not passing through p induces an oval $\mathcal{O}$ in $\mathcal{P}_p$. Since this circle also represents a circle of $\mathcal{L}'_p$, we just obtain the dual oval $\mathcal{O}^*$ of $\mathcal{O}$ in $\mathcal{P}'_{\overline{\infty}}$. Hence, the whole process involves forming the dual of the derived projective plane $\mathcal{P}_p$ plus all duals of the ovals in $\mathcal{P}_p$ that are induced by circles of $\mathcal{L}$; we then remove one line to obtain the affine part of the sister $\mathcal{L}'_p$ and add one parallel class at infinity in order to complete the Laguerre plane.

THEOREM 3 *Let $\mathcal{L}$ be a 4-dimensional non-classical elation Laguerre plane and let p be a point of $\mathcal{L}$. Then the sister $\mathcal{L}'_p$ of $\mathcal{L}$ with respect to p is a 4-dimensional Laguerre plane that is not an elation Laguerre plane; the derived plane of $\mathcal{L}'_p$ at $p_{\overline{\infty}}$, the point of $\mathcal{L}'_p$ that corresponds to $\overline{\infty}$ in the Lie geometry of $\mathcal{L}$, is a 4-dimensional translation plane. All other derived planes at points of the parallel class of $p_{\overline{\infty}}$ are shift planes.*
Each automorphism of $\mathcal{L}'_p$ fixes $p_{\overline{\infty}}$ and the dimension of the automorphism group of $\mathcal{L}'_p$ equals the dimension of the stabilizer of p in the automorphism group of $\mathcal{L}$.

Proof. The elations of $\mathcal{L}$ that fix p induce translations of $\mathcal{L}'_p$ in the derived plane at $p_{\overline{\infty}}$. Hence the derived plane of $\mathcal{L}'_p$ at $p_{\overline{\infty}}$ is a translation plane. At points of $|p_{\overline{\infty}}|$ different from $p_{\overline{\infty}}$ the 4-dimensional elation group fixing p induces a 4-dimensional abelian group of automorphisms. Since not all of them are translations we obtain a 4-dimensional shift group and the derived plane is a shift plane, compare [10], section 74.
Suppose that $\mathcal{L}'_p$ is an elation Laguerre plane. Then each derived plane of $\mathcal{L}'_p$ is a dual translation plane. In particular, the derived plane of $\mathcal{L}'_p$ at $p_{\overline{\infty}}$ is a dual translation plane and also a translation plane and thus Desarguesian by [10], Theorem 72.11. Hence $\mathcal{L}'$ must be classical by [8], Corollary 2.5. However, the associated generalized quadrangle of the classical complex Laguerre plane is the classical symmetric generalized quadrangle $Q(4, \mathbb{C})$ (the dual of the classical symplectic generalized quadrangle $\mathrm{W}(\mathbb{C})$) so that every sister of the classical complex Laguerre plane is also classical. This contradicts our assumption that $\mathcal{L}$ is non-classical.
It is clear that every automorphism of $\mathcal{L}$ that fixes p induces an automorphism of $\mathcal{L}'_p$ that fixes $p_{\overline{\infty}}$. Thus the dimension of the automorphism group Γ' of $\mathcal{L}'_p$ is at least as big as the dimension of the stabilizer Γ_p of p in the automorphism group Γ of $\mathcal{L}$. Furthermore, Γ' is transitive on the affine points of $\mathcal{L}'_p$ (those not on the parallel class π containing $p_{\overline{\infty}}$). Suppose that Γ' does not fix $p_{\overline{\infty}}$. If $p_{\overline{\infty}}$ is moved to a different point q on π, this means that there is an automorphism of the common associated generalized quadrangle $\mathcal{Q}$ that maps $\overline{\infty}$ to q. But in $\mathcal{L}$ we see that there also is an automorphism of $\mathcal{Q}$ that takes q to p. Hence $\mathcal{L}$ and $\mathcal{L}'_p$ are isomorphic by [14], Theorem 3.14, which contradicts $\mathcal{L}'_p$ not being an elation Laguerre plane. If $p_{\overline{\infty}}$ can be taken to a point not on π, then $p_{\overline{\infty}}$ has a 4-dimensional orbit. Such an orbit is open in the 4-dimensional point space of $\mathcal{L}'_p$ and thus must contain points of π different from $p_{\overline{\infty}}$. Hence we are in the former case and obtain a contradiction as before. This shows that $p_{\overline{\infty}}$ is fixed by every automorphism of $\mathcal{L}'_p$ and $\dim \Gamma' = \dim \Gamma_p$. □

The method as outlined at the beginning of this section can be applied to any 4-dimensional elation Laguerre plane and any of its points. Note that points that are in the same orbit under the automorphism group of the Laguerre plane yield isomorphic sisters. From the 4-dimensional elation Laguerre planes of group dimension 10 we can obtain in this fashion 4-dimensional Laguerre planes of group dimensions 8, 7 or 6, none of which is an elation Laguerre plane.

We shall carry out this procedure for the single 4-dimensional elation Laguerre plane that has a solvable 10-dimensional automorphism group and the semi-classical 4-dimensional Laguerre planes. Due to the pasted form of circles in the $\mathrm{SL}_2(\mathbb{R})$-elation Laguerre plane, see Example 2, which causes $D(z)$ to be not differentiable, and due to the lack of apparent symmetries, only the circles of sisters obtained from the other 4-dimensional elation Laguerre plane of group dimension 10 seem to have nice descriptions on the affine part.
Of course, one can also form the sister of a Laguerre plane $\mathcal{L}$ with respect to a circle of $\mathcal{L}$. However, the circles of sisters obtained in this way do not have as nice a description as with respect to points and results about the group dimensions of these sisters are not so easy to obtain even in the case of elation Laguerre planes. The same difficulties arise if one wants to form the sister of a sister of $\mathcal{L}$ since a point in the sister of $\mathcal{L}$ may correspond to a circle of $\mathcal{L}$.
Specifically, we are using the point $p = (\infty, 0)$ in order to form sisters in the following sections. Then, if $\mathcal{L}$ is represented by matrix-valued maps A and B as in Theorem 1, the derived affine plane $\mathcal{A}_p$ of $\mathcal{L}$ at p has the non-vertical lines $\{(z, c^2B(z) + c^3) \mid z \in \mathbb{R}^2\}$ for $c^2, c^3 \in \mathbb{R}^2$. Under dualisation the line at infinity of $\mathcal{P}_p$ becomes the infinite point ω of $\mathcal{P}'_p$ and the point ω in $\mathcal{P}_p$ becomes the line at infinity of $\mathcal{P}'_p$. Parallel lines of $\mathcal{A}_p$ (that is, with the same parameter c^2) give rise to parallel points in $\mathcal{L}'_p$. Introducing coordinates in $\mathcal{L}'_p$ such that a point (z, w) of $\mathcal{L}'_p$ corresponds to the circle $C_{(0,z,-w)}$ in $\mathcal{L}$, the point set of the affine part of $\mathcal{L}'_p$ is $\mathbb{R}^4$ with two points (z, w) and (z', w') being parallel if and only if $z = z'$. The non-vertical lines of $\mathcal{A}'_{p_\infty}$ (that is, circles of $\mathcal{L}'_p$ through p_∞) are the sets

$$\{(z, z \cdot B(m) + t) \mid z \in \mathbb{R}^2\}$$

for $m, t \in \mathbb{R}^2$ with the same matrix-valued map B as for $\mathcal{L}$. Conversely, a line with parameters m and t in the derived affine plane of $\mathcal{L}'_p$ at p_∞ corresponds to the point $(m, -t)$ in $\mathcal{L}$. We are making these conventions in the following sections.
Note that an elation of $\mathcal{L}$ fixing p given by $(z, w) \mapsto (z, w + d^2B(z) + d^3)$ for $d^2, d^3 \in \mathbb{R}^2$ gives rise to the translation $(z, w) \mapsto (z+d^2, w-d^3)$ on the affine part of $\mathcal{L}'_p$. Furthermore, because the elation fixes the parallel class of p pointwise, the same is true for the translation of $\mathcal{L}'_p$ and the parallel class at infinity, that is, the parallel class of p_∞. In particular, this implies that we only have to dualise the circles given in $\mathcal{A}_p$ by $\{(z, c^1A(z)) \mid z \in \mathbb{R}^2\}$ for $c^1 \in \mathbb{R}^2$, $c^1 \neq 0$. All other circles of $\mathcal{L}'_p$ are then obtained from these by applying translations where a circle and its translate pass through the same point on the parallel class at infinity.
The dualisation of the circle with equation $w = c^1A(z)$ for $c^1 \in \mathbb{R}^2$, $c^1 \neq 0$, yields a planar map generating the shift plane that occurs as derived plane at the point determined by c^1 on the parallel class at infinity; see [10], 74.1 and 74.3, for a definition and characterization of planar maps. In case of a differentiable spread map B, the planar maps are also differentiable. Not surprisingly, these planar maps are related to the spread map B. So let $\mathcal{L}$ be a 4-dimensional non-classical elation Laguerre plane with admissible parametrization D and assume that D is differentiable on $\mathbb{R}^2$. Let $\mathcal{L}'$ be the sister of $\mathcal{L}$ with respect to the point $(\infty, 0)$. Then the 4-dimensional shift planes that occur as derived planes of $\mathcal{L}'$ at points on the parallel class at infinity different from p_∞ are integral planes of the

translation plane that occurs for $p_{\overline{\infty}}$, that is, the planar functions describing these shift planes are integrals of the spread of the translation plane; see [10], 74.14-16, for a definition of integrals of spreads and planes. This is a direct consequence of coherence properties of Laguerre planes, in particular, K1-coherence which implies that 'touching is the limit of proper intersection', compare [5], section 3, or [17], section 5.
Note that the differentiability of B on $\mathbb{R}^2$ suffices, because if $B(z)$ is differentiable at z_0 then so is $A(z)$, see [17], Proposition 5.10. The above statement on integrals also carries over to regions of differentiability of B in case B is not differentiable everywhere as, for example, in the semi-classical 4-dimensional Laguerre planes.

4 A sister of the 4-dimensional elation Laguerre plane with solvable 10-dimensional automorphism group

Throughout this section $\mathcal{L}$ denotes the unique 4-dimensional elation Laguerre plane admitting a solvable 10-dimensional automorphism group as given in Example 2 by the matrix valued maps $A(z)$ and $B(z)$. We carry out the construction of the previous section for $\mathcal{L}$ and the point $(\infty, 0)$. This leaves us with a particularly large group of automorphisms of the new Laguerre plane $\mathcal{L}' = \mathcal{L}'_{(\infty,0)}$. We begin with a summary of properties of the Laguerre plane $\mathcal{L}$, see [21] for details.

THEOREM 4 *Let $\mathcal{L}$ be the unique 4-dimensional elation Laguerre plane admitting a solvable 10-dimensional automorphism group.*
The transformations $(z, w) \mapsto (z, w + c \cdot D(z))$ for $c \in \mathbb{R}^6$ are continuous automorphisms of $\mathcal{L}$. A point (∞, w) on the parallel class at infinity is taken to $(\infty, w + c^1)$ where $c = (c^1, c^2, c^3)$.
The transformations $\gamma_{r,d,s,t}$ given by

$$\gamma_{r,d,s,t} : (z, w) \mapsto \begin{cases} \left(z \cdot \begin{pmatrix} d & ds \\ 0 & d^2 \end{pmatrix} - (s, \frac{1}{2}s^2 + t), w \cdot \begin{pmatrix} r & rs \\ 0 & rd \end{pmatrix} \right), & \text{for } z \in \mathbb{R}^2, \\ \left(\infty, w \cdot \begin{pmatrix} rd^{-4} & rsd^{-4} \\ 0 & rd^{-3} \end{pmatrix} \right), & \text{for } z = \infty, \end{cases}$$

where $d, r, s, t \in \mathbb{R}$, $d, r \neq 0$ are continuous automorphisms of $\mathcal{L}$. A circle C_c, $c \in \mathbb{R}^6$, is mapped to a circle whose coordinate vector is $c \cdot M_{r,d,s,t}$ where $M_{r,d,s,t}$ is the 6×6 matrix

$$M_{r,d,s,t} = \frac{r}{d^4} \begin{pmatrix} 1 & s & \frac{1}{2}s^2 + t & s(\frac{1}{6}s^2 + t) & \frac{1}{24}s^4 + \frac{1}{2}s^2 t + \frac{1}{2}t^2 & s(\frac{1}{120}s^4 + \frac{1}{6}s^2 t + \frac{1}{2}t^2) \\ 0 & d & ds & d(\frac{1}{2}s^2 + t) & ds(\frac{1}{6}s^2 + t) & d(\frac{1}{24}s^4 + \frac{1}{2}s^2 t + \frac{1}{2}t^2) \\ 0 & 0 & d^2 & d^2 s & d^2(\frac{1}{2}s^2 + t) & d^2 s(\frac{1}{6}s^2 + t) \\ 0 & 0 & 0 & d^3 & d^3 s & d^3(\frac{1}{2}s^2 + t) \\ 0 & 0 & 0 & 0 & d^4 & d^4 s \\ 0 & 0 & 0 & 0 & 0 & d^5 \end{pmatrix}.$$

Furthermore, the group $\Sigma = \{\gamma_{r,d,s,t} \mid r,d,s,t \in \mathbb{R}, r,d \neq 0\}$ *fixes the circle* C_0 *and is transitive on* $C_0 \setminus \{(\infty,0)\}$.
Two circles C_c *and* C_d *touch each other at a point* (z_0, w_0) *for* $z_0 \in \mathbb{R}$ *if and only if the coordinate vectors* $c, d \in \mathbb{R}^6$ *satisfy* $c \cdot (D_y(z_0), D(z_0)) = d \cdot (D_y(z_0), D(z_0))$, *that is, c and d satisfy*

$$c \cdot \begin{pmatrix} B(z_0) & A(z_0) \\ I_2 & B(z_0) \\ 0 & I_2 \end{pmatrix} = d \cdot \begin{pmatrix} B(z_0) & A(z_0) \\ I_2 & B(z_0) \\ 0 & I_2 \end{pmatrix}.$$

(Note that $A_y(z) = B(z)$, $B_y(z) = I_2$ *and* $D_x(z) = D_y(z)B_x(z)$.*)*

The derived plane of $\mathcal{L}$ at $(\infty, 0)$ is the *dual Betten plane* $\mathcal{B}'$, that is, the dual of the unique 4-dimensional translation plane $\mathcal{B}$, discovered by D. Betten, that has a solvable 8-dimensional collineation group, see [1] or [10], §73, for this plane, which we simply refer to as the *Betten plane*. (Finite analogues of $\mathcal{B}$ were discovered independently by M. Walker [22]. Since we are in a topological context we use the term Betten plane instead of Betten-Walker plane.)
We first look at the affine part and determine later how the extra parallel class fits in. Dualising the derived plane at $(\infty, 0)$ we obtain, of course, the Betten plane $\mathcal{B}$. In the next step we 'dualise' the circle $C_{(0,1,0,0,0,0)}$ of $\mathcal{L}$. Let $e_1 = (1,0)$ and $e_2 = (0,1) \in \mathbb{R}^2$ be the standard basis vectors for $\mathbb{R}^2$. Then $C_{(0,1,0,0,0,0)} = \{(z, e_2A(z)) \mid z \in \mathbb{S}_2\}$ and the tangent circle C_c to $C_{(0,1,0,0,0,0)}$ at $(z, e_2A(z))$ for $z \in \mathbb{R}^2$ that passes through $(\infty, 0)$ is determined from the system of linear equations

$$c \cdot (D(\infty), D_y(z), D(z)) = (0, e_2A_y(z), e_2A(z))$$

where D_y and A_y denote the partial derivatives with respect to y; compare Theorem 4. But $A_y(z) = B(z)$ and $B_y(z) = I_2$ so that

$$c \cdot \begin{pmatrix} I_2 & B(z) & A(z) \\ 0 & I_2 & B(z) \\ 0 & 0 & I_2 \end{pmatrix} = (0, e_2B(z), e_2A(z)) = (0, z, e_2A(z)).$$

From the system of linear equations above we obtain for $c = (c^1, c^2, c^3)$ that

$$\begin{aligned} c^1 &= 0, \\ c^2 &= z, \\ c^3 &= e_2A(z) - zB(z). \end{aligned}$$

Passing over to the coordinates of $\mathcal{L}'$ (where z becomes a parameter, c^2 becomes z and c^3 becomes $-w$) one finds the set

$$\{(z, zB(z) - e_2A(z)) \mid z \in \mathbb{R}^2\} = \left\{(x, y, xy + \frac{1}{3}x^3, \frac{1}{2}y^2 - \frac{1}{12}x^4) \mid x, y \in \mathbb{R}\right\}. \qquad (1)$$

This is equivalent to Knarr's surface as given in [3] or [10], 74.24, under the transformation $(z,w) \mapsto (-z,w)$. Each automorphism in the 8-dimensional stabilizer $\Gamma_{(\infty,0)}$ of $(\infty,0)$ in the automorphism group of $\mathcal{L}$ induces an automorphism of $\mathcal{L}'$. From Theorem 4 we obtain the transformations

$$(z,w) \mapsto (z+a, w+b)$$

for $a, b \in \mathbb{R}^2$ (translations) and

$$(z,w) \mapsto (z,w) \cdot \tilde{M}_{r,d,s,t}$$

for $d, r, s, t \in \mathbb{R}$, $d, r \neq 0$, where $\tilde{M}_{r,d,s,t}$ denotes the lower right 4×4 block in the matrix $M_{r,d,s,t}$ as in Theorem 4. (That is, $(z,w) \cdot \tilde{M}_{r,d,s,t}$ is the projection of $(0,z,w) \cdot M_{r,d,s,t}$ onto the last four coordinates.) Since by Theorem 4 the circle $C_{(e_2,0,0)}$ can be taken by automorphisms in $\Gamma_{(\infty,0)}$ to any circle $C_{(re_2,c^2,c^3)}$ for $r \in \mathbb{R}$, $r \neq 0$, $c^2, c^3 \in \mathbb{R}^2$, we obtain the circles in $\mathcal{L}'$ corresponding to $C_{(re_2,c^2,c^3)}$ in $\mathcal{L}$ by applying the transformations above to the set (1). (It suffices to use the transformations $(z,w) \mapsto (\frac{1}{r}z + a, \frac{1}{r}w + b)$.) In this way one finds the 5-parametric family of sets

$$\{(x, y, r(x-a_1)(y-a_2) + \frac{r^2}{3}(x-a_1)^3 + b_1, \frac{r}{2}(y-a_2)^2 - \frac{r^3}{12}(x-a_1)^4 + b_2) \mid x, y \in \mathbb{R}\}$$

for $a_2, a_2, b_1, b_2, r \in \mathbb{R}$, $r \neq 0$.
Since $r = 1$ yields essentially Knarr's surface and all its translates, one obtains a copy of basically the unique 4-dimensional shift plane of group dimension 7, compare [10], Theorem 74.27. All these circles should pass through the same point on the parallel class at infinity so that the derived plane at this point is isomorphic to this shift plane.
Similarly, a circle C_c of $\mathcal{L}$ with $c = (c_1, \ldots, c_6)$, $c_1 \neq 0$, can be obtained from $C_{(-3e_1,0,0)}$ by applying automorphisms in $\Gamma_{(\infty,0)}$. It therefore suffices to 'dualise' $C_{(-3e_1,0,0)}$ and then apply the corresponding group in $\mathcal{L}'$. As before one obtains the system of linear equations

$$c \cdot \begin{pmatrix} I_2 & B(z) & A(z) \\ 0 & I_2 & B(z) \\ 0 & 0 & I_2 \end{pmatrix} = (0, -3e_1B(z), -3e_1A(z))$$

which now yields

$$\begin{aligned} c^1 &= 0, \\ c^2 &= -3e_1B(z), \\ c^3 &= -3e_1(A(z) - B(z)^2). \end{aligned}$$

Passing over again to the coordinates of $\mathcal{L}'$ one finds the parametric representation

$$\begin{aligned} z &= (-3\beta - 3\alpha^2, \alpha^3), \\ w &= (-\frac{3}{2}\beta^2 - 3\alpha^2\beta - \frac{3}{4}\alpha^4, \alpha^3\beta + \frac{2}{5}\alpha^5), \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$. Eliminating α and β from the first of these two equations yields

$$\begin{aligned} \alpha &= \sqrt[3]{y}, \\ \beta &= -\sqrt[3]{y^2} - \frac{1}{3}x, \end{aligned}$$

so that we obtain the set

$$\left\{ (x, y, \frac{3}{4}y^{4/3} - \frac{1}{6}x^2, -\frac{3}{5}y^{5/3} - \frac{1}{3}xy) \mid x, y \in \mathbb{R} \right\}$$

as the affine part of a circle in $\mathcal{L}'$. Applying translations and the transformations

$$(x, y, u, v) \mapsto (rx, ry + rsx, ru - rsy - \frac{1}{2}rs^2x, rv + rsu - \frac{1}{2}rs^2y - \frac{1}{6}rs^3x)$$

yields the remaining circles.

THEOREM 5 *The sets*

- $\{(x, y, (l + k^2)x + ky + b_1, -\frac{1}{3}k^3x + ly + b_2) \mid x, y \in \mathbb{R}\} \cup \{(\infty, (0, 0))\}$
  for $b_1, b_2, k, l \in \mathbb{R}$*;*
- $\{(x, y, u_r(x - a_1, y - a_2) + b_1, v_r(x - a_1, y - a_2) + b_2) \mid x, y \in \mathbb{R}\} \cup \{(\infty, (0, r))\}$
  for $a_1, a_2, b_1, b_2, r \in \mathbb{R}$*,* $r \neq 0$*, where*

$$\begin{aligned} u_r(x, y) &= rxy + \frac{r^2}{3}x^3, \\ v_r(x, y) &= \frac{r}{2}y^2 - \frac{r^3}{12}x^4; \end{aligned}$$

- $\{(x, y, u_{r,s}(x - a_1, y - a_2) + b_1, v_{r,s}(x - a_1, y - a_2) + b_2) \mid x, y \in \mathbb{R}\}$
  $\cup\{(\infty, (-\frac{r}{3(s^2+1)}, -\frac{rs}{3(s^2+1)}))\}$
  for $a_1, a_2, b_1, b_2, r, s \in \mathbb{R}$*,* $r \neq 0$*, where*

$$\begin{aligned} u_{r,s}(x, y) &= \frac{3r^{1/3}}{4}(y - sx)^{4/3} - sy - \frac{r}{6}x^2 + \frac{s^2}{2}x, \\ v_{r,s}(x, y) &= -\frac{3r^{2/5}}{5}(y - sx)^{5/3} + \frac{3r^{1/3}s}{4}(y - sx)^{4/3} - \frac{r}{3}xy - \frac{s^2}{2}y + \frac{rs}{6}x^2 + \frac{s^3}{3}x. \end{aligned}$$

form the circles of a 4-dimensional Laguerre plane $\mathcal{L}'$ *on* $(\mathbb{R}^2 \cup \{\infty\}) \times \mathbb{R}^2$*. Two points* (z, w) *and* (z', w') *are parallel if and only if* $z = z'$*.* $\mathcal{L}'$ *has group dimension 8 and is not an elation Laguerre plane. The derived plane of* $\mathcal{L}'$ *at* $(\infty, 0)$ *is a 4-dimensional translation plane, the Betten plane* $\mathcal{B}$*, and the derived planes of* $\mathcal{L}'$ *at* (∞, w)*,* $w \neq 0$*, are 4-dimensional shift planes that are integrals of* $\mathcal{B}$*.*

Proof. The form of the affine parts of circles has been derived above. In order to determine the corresponding points on the parallel class π_∞ at infinity one has to identify those circles that pass through the same point on π_∞; the actual labels these points receive is somewhat arbitrary. The only restriction is that points close to p in $\mathcal{L}$ should get coordinates (∞, w) in $\mathcal{L}'$ with 'large' w. By convention for this kind of model of a Laguerre plane the first set of circles, which form the non-vertical lines in $\mathcal{B}$, should pass through $(\infty, (0,0))$. We use the homeomorphism $\varphi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$ given by $\varphi(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. (In complex coordinates this is just the map $z \mapsto 1/\bar{z}$. This map extends to a homeomorphism of the 2-sphere $\mathbb{S}_2$ which interchanges the points ∞ and $(0,0)$ so that the restriction above is satisfied.) A point with coordinates (∞, w) in $\mathcal{L}$ is assigned the coordinates $(\infty, \varphi(w))$ in $\mathcal{L}'$.

Since translations $(z,w) \mapsto (z+a, w+b)$ of $\mathcal{L}'$ come from elations of $\mathcal{L}$ that fix p, we see that each translation fixes every point of π_∞. Hence a circle and its translate pass through the same point on π_∞. The automorphisms $\gamma_{r,1,s,0}$ of $\mathcal{L}$ from Theorem 4 for $r, s \in \mathbb{R}$, $r \neq 0$, form a group that has three orbits on the parallel class at infinity of $\mathcal{L}$, one fixed point, one 1-dimensional orbit and one 2-dimensional orbit. Furthermore, this group is sharply transitive on the 2-dimensional orbit and so is the subgroup $\{\gamma_{r,1,0,0} \mid r \in \mathbb{R}, r \neq 0\}$ on the 1-dimensional orbit. Applying $\gamma_{1/r,1,0,0}$ to the base circle $C_{(e_2,0,0)}$ of $\mathcal{L}$ yields in $\mathcal{L}'$ a circle of the second kind with parameter r. Hence these circles pass through $(\infty, \varphi(0, 1/r)) = (\infty, (0, r))$. Similarly, applying $\gamma_{1/r,s,0,0}$ to the base circle $C_{(-3e_1,0,0)}$ of $\mathcal{L}$ yields in $\mathcal{L}'$ a translate of a circle of the second kind with parameters r and s. Hence these circles pass through $(\infty, \varphi(-3/r, -3s/r)) = (\infty, (-\frac{r}{3(s^2+1)}, -\frac{rs}{3(s^2+1)}))$.

That $\mathcal{L}'$ is no elation Laguerre plane and the group dimension of $\mathcal{L}'$ follow from Theorem 3. It is clear by construction that the derived plane of $\mathcal{L}'$ at $(\infty, 0)$ is the Betten plane $\mathcal{B}$. □

As mentioned before, points that are in the same orbit under the automorphism group of the Laguerre plane yield isomorphic sisters. For our Laguerre plane $\mathcal{L}$ we therefore obtain precisely two non-isomorphic sisters, the Laguerre plane $\mathcal{L}'$ from Theorem 5 above and the sister plane $\mathcal{L}'' = \mathcal{L}'_{(0,0)}$ obtained from $\mathcal{L}$ with respect to the affine point $(0,0)$. Since the latter plane has group dimension 6 by Theorem 3, the Laguerre planes $\mathcal{L}$, $\mathcal{L}'$ and $\mathcal{L}''$ are mutually non-isomorphic. Again, the latter plane $\mathcal{L}''$ does not have as nice a representation as $\mathcal{L}'$.

5 Sisters of semi-classical Laguerre planes

Throughout this section $\mathcal{L} = \mathcal{L}_q$ denotes a semi-classical Laguerre plane where $0 < q < 1$ is the plane parameter. For a fixed q we carry out the construction of section 3 for $\mathcal{L}$ and the point $(\infty, 0)$. This leaves us again with a particularly large group of automorphisms of the new Laguerre plane $\mathcal{L}' = \mathcal{L}'_{(\infty,0)}$.

In order to simplify the computations we drop the normalisations made for the second row of $B(z)$ in Theorem 1 and introduce new circle coordinates via the transformation

$$(c_1, c_2, c_3, c_4, c_5, c_6) \mapsto (-c_1, -c_2, c_4, -c_3, c_5, c_6).$$

This means that the matrices $A(z)$ and $B(z)$ as in Example 2 are replaced for $z \in \mathbb{R}^2$ by $-A(z)$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B(z)$, respectively, that is, the new matrices, which we label again $A(z)$ and $B(z)$ for simplicity, are

$$\begin{aligned} A(z) &= \begin{pmatrix} x^2 - yh(y) & 2xy \\ -2xh(y) & x^2 - yh(y) \end{pmatrix}, \\ B(z) &= \begin{pmatrix} x & y \\ -h(y) & x \end{pmatrix}, \end{aligned}$$

where h denotes the homeomorphism of $\mathbb{R}$ given by

$$h(y) = \begin{cases} y, & \text{for } y \geq 0, \\ qy, & \text{for } y \leq 0. \end{cases}$$

Then $A(z) = B(z)^2$ for all $z \in \mathbb{R}^2$ and circles are of the form

$$C_{(c^1,c^2,c^3)} = \left\{ (z, c^1 B(z)^2 + c^2 B(z) + c^3) \mid z \in \mathbb{R}^2 \right\} \cup \left\{ (\infty, c^1) \right\}$$

for $c^1, c^2, c^3 \in \mathbb{R}^2$. The matrix-valued map B represents a 4-dimensional translation plane, namely one of the planes with reducible $\mathrm{SL}_2(\mathbb{R})$-action on the translation group, see [10], Theorem 73.13.

We dualise the circle $C_{(d,0,0)}$ of $\mathcal{L}$ for $d = (r, s) \in \mathbb{R}^2$, $d \neq 0$. Similar to the previous section the tangent circle C_c to $C_{(d,0,0)}$ at $(z, d \cdot A(z))$ for $z \in \mathbb{R}^2$ that passes through $(\infty, 0)$ is determined from the system of linear equations

$$c \cdot (D(\infty), D_x(z), D(z)) = (0, d \cdot A_x(z), d \cdot A(z)).$$

Note that we are using the partial derivatives with respect to x instead of the ones with respect to y. This is due to the changes we made to the matrices $A(z)$ and $B(z)$ at the beginning of this section. One finds that $A_x(z) = 2B(z)$ and $B_x(z) = I_2$ so that the system of linear equations above becomes

$$c \cdot \begin{pmatrix} I_2 & 2B(z) & B(z)^2 \\ 0 & I_2 & B(z) \\ 0 & 0 & I_2 \end{pmatrix} = (0, 2d \cdot B(z), d \cdot B(z)^2).$$

Hence for $c = (c^1, c^2, c^3)$ we have

$$\begin{aligned} c^1 &= 0, \\ c^2 &= 2d \cdot B(z), \\ c^3 &= d \cdot B(z)^2. \end{aligned}$$

Passing over to the coordinates of $\mathcal{L}'$ (where z becomes a parameter t, c^2 becomes z and c^3 becomes w) one finds the parametric description

$$\begin{aligned} z &= 2d \cdot B(t), \\ w &= \frac{1}{2} z \cdot B(t). \end{aligned}$$

In order to eliminate $t = (t_1, t_2)$ from the first equation, note that

$$\frac{1}{2}(ry - sx) = r^2 t_2 + s^2 h(t_2) = \begin{cases} (dr^2 + s^2)t_2, & \text{if } t_2 \geq 0, \\ (dr^2 + qs^2)t_2, & \text{if } t_2 < 0. \end{cases}$$

Since t_2 and $ry - sx$ have the same sign, one finds that

$$t_2 = \begin{cases} \frac{ry - sx}{2(r^2+s^2)}, & \text{for } ry - sx \geq 0, \\ \frac{ry - sx}{2(r^2+qs^2)}, & \text{for } ry - sx < 0. \end{cases}$$

Then

$$t_1 = \begin{cases} \frac{rx+sy}{2(r^2+s^2)}, & \text{for } ry - sx \geq 0, \\ \frac{rx+sy}{2(r^2+s^2)} - \frac{(1-q)rs(ry-sx)}{2(r^2+qs^2)}, & \text{for } ry - sx < 0. \end{cases}$$

Substitution into the equation for w and straightforward algebraic manipulation then yields

$$w = \begin{cases} \frac{1}{4(r^2+s^2)}(r(x^2 - y^2) + 2sxy, 2rxy - s(x^2 - y^2)), & \text{for } ry - sx \geq 0, \\ \frac{1}{4(r^2+qs^2)}(r(x^2 - qy^2) + 2qsxy, 2rxy - s(x^2 - qy^2)), & \text{for } ry - sx < 0. \end{cases}$$

Using the substitution

$$\begin{aligned} \tilde{r} &= \frac{r}{4(r^2 + s^2)} \\ \tilde{s} &= -\frac{s}{4(r^2 + s^2)} \end{aligned}$$

this finally becomes

$$w = \begin{cases} (\tilde{r}(x^2 - y^2) - 2\tilde{s}xy, 2\tilde{r}xy + \tilde{s}(x^2 - y^2)), & \text{for } \tilde{r}y + \tilde{s}x \geq 0, \\ \frac{\tilde{r}^2+\tilde{s}^2}{\tilde{r}^2+q\tilde{s}^2}(\tilde{r}(x^2 - qy^2) - 2q\tilde{s}xy, 2\tilde{r}xy + \tilde{s}(x^2 - qy^2)), & \text{for } \tilde{r}y + \tilde{s}x < 0. \end{cases}$$

In particular, for $\tilde{s} = 0$ one obtains $w = \tilde{r}(x^2 - yh(y), 2xy))$, which describes a complex skew-parabola, compare [2].
Since the transformation $(r, s) \mapsto (\tilde{r}, \tilde{s})$ is a homeomorphism of $\mathbb{R}^2 \setminus \{(0, 0)\}$ that extends to a homeomorphism of $\mathbb{S}_2$ that interchanges $(0, 0)$ and ∞, we see that we can extend the affine part of the dualised circle by the point $(\infty, (\tilde{r}, \tilde{s}))$ on the parallel class at infinity in $\mathcal{L}'$. All other circles are obtained by forming translates. In particular, for $\tilde{s} = 0$ and $\tilde{r} \neq 0$ fixed the resulting circles form the lines of a complex skew-parabola, compare [2] or [10], Theorem 74.30. Hence these 4-dimensional shift planes occur as derived planes of $\mathcal{L}'$. The same is true, up to isomorphism, for the other derived planes at points on the parallel class at infinity.

THEOREM 6 *The sets*

- $\{(x, y, kx - h(l)y + b_1, lx + ky + b_2) \mid x, y \in \mathbb{R}\} \cup \{(\infty, (0, 0))\}$ *for* $b_1, b_2, k, l \in \mathbb{R}$*;*
- $\{(x, y, u_{r,s}(x - a_1, y - a_2) + b_1, v_{r,s}(x - a_1, y - a_2) + b_2) \mid x, y \in \mathbb{R}\} \cup \{(\infty, (r, s))\}$ *for* $a_1, a_2, b_1, b_2, r, s \in \mathbb{R}$, $(r, s) \neq (0, 0)$, *where*

$$u_{r,s}(x, y) = \begin{cases} r(x^2 - y^2) - 2sxy, & \text{for } ry + sx \geq 0 \\ \frac{r^2+s^2}{r^2+qs^2}(r(x^2 - qy^2) - 2qsxy), & \text{for } ry + sx < 0 \end{cases}$$

$$v_{r,s}(x, y) = \begin{cases} 2rxy + s(x^2 - y^2), & \text{for } ry + sx \geq 0 \\ \frac{r^2+s^2}{r^2+qs^2}(2rxy + s(x^2 - qy^2)) & \text{for } ry + sx < 0 \end{cases}$$

form the circles of a 4-dimensional Laguerre plane $\mathcal{L}'$ *on* $(\mathbb{R}^2 \cup \{\infty\}) \times \mathbb{R}^2$*. Two points* (z, w) *and* (z', w') *are parallel if and only if* $z = z'$*.* $\mathcal{L}'$ *has group dimension 7 and is not an elation Laguerre plane. The derived plane of* $\mathcal{L}'$ *at* $(\infty, 0)$ *is a 4-dimensional translation plane admitting a reducible* $\mathrm{SL}_2(\mathbb{R})$*-action on the translation group and derived planes at* (∞, w) *for* $w \neq 0$ *are isomorphic to complex skew-parabola planes.*

Note that the definitions of the circles in Theorem 6 remain valid for $q = 1$ in which case one just obtains the classical complex Laguerre plane. In particular, the equation $w = (r(x^2 - y^2) - 2sxy, 2rxy + s(x^2 - y^2))$ describes a circle of the classical complex Laguerre plane. Furthermore, the set

$$\{(x, y, r(x^2 - qy^2) - 2qsxy, 2rxy + s(x^2 - qy^2)) \mid x, y \in \mathbb{R}^2\}$$

is taken under the transformation $(x, y, u, v) \mapsto (x, \sqrt{q}y, u, \sqrt{q}v))$ to the set

$$\{(x, y, r(x^2 - y^2) - 2\sqrt{q}sxy, 2rxy + \sqrt{q}s(x^2 - y^2)) \mid x, y \in \mathbb{R}^2\},$$

which is again a circle of the classical complex Laguerre plane. Hence circles for the sister $\mathcal{L}'$ are made up of two halves of classical circles. However, the pasting process in $\mathcal{L}'$ is rather different from the one in semi-classical 4-dimensional Laguerre planes where circles are pasted together along a fixed 3-dimensional separating set in the point space.

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