A characterization of certain elation Laguerre planes in terms of Kleinewillinghöfer types *

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Abstract

We characterize the non-classical 4-dimensional elation Laguerre planes as precisely those 4-dimensional Laguerre planes of Kleinewillinghöfer type I.D.1. Furthermore, in the class of 2- or 4-dimensional Laguerre planes or finite Laguerre planes of odd order, the non-miquelian elation Laguerre planes are precisely the Laguerre planes of Kleinewillinghöfer type D.

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1 Introduction

Elation Laguerre planes are characterized by the existence of a group of automorphisms that acts trivially on the set of parallel classes and regularly on the set of circles. They can be described in terms of dual pseudo-ovals thus generalizing ovoidal Laguerre planes; see [13] and [8]. Knarr [11] further characterized elation Laguerre planes geometrically by a certain degeneration, **M2**, of Miquel's configuration. In fact, Bröcker [1] showed that **M2** is the only degeneration of Miquel's configuration on eight or seven points that leads to a non-miquelian Laguerre plane. In that respect elation Laguerre planes are the closest thing one can get to miquelian Laguerre planes. For a 4-dimensional Laguerre plane to be an elation Laguerre plane it suffices that the kernel is large enough or is transitive on the set of circles; see [22] or [24], Theorem 2.7. Furthermore, the first examples of non-classical 4-dimensional Laguerre planes found were elation laguerre planes. All of this shows that elation Laguerre planes form a nice and important subclass of Laguerre planes.

In this paper we investigate the possible Kleinewillinghöfer types of 4-dimensional elation Laguerre planes, see [9], [16] or section 3 for these types, and show that type D characterizes non-classical elation Laguerre planes among 4-dimensional Laguerre planes. This extends an analogous characterization of ovoidal 2-dimensional Laguerre planes; see

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[17], Corollary 4.5. More precisely, we show that the non-classical 4-dimensional elation Laguerre planes are exactly the 4-dimensional Laguerre planes of Kleinewillinghöfer type I.D.1.

2 $2n$-dimensional elation Laguerre planes

A *$2n$-dimensional Laguerre plane*, $n = 1, 2$, is a Laguerre plane $\mathcal{L} = (P, \mathcal{C}, ||)$ whose point set P and circle set $\mathcal{C}$ carry Hausdorff topologies such that P is $2n$-dimensional locally compact and such that the geometric operations of joining three points by a circle, of intersecting two different circles, parallel projection and touching are continuous with respect to the induced topologies on their respective domains of definition. In this case the circle set $\mathcal{C}$ is homeomorphic to $\mathbb{R}^{3n}$, and elements of $\mathcal{C}$ are homeomorphic to the n-sphere $\mathbb{S}_n$. For more information about topological Laguerre planes we refer to [5], [6], [16] and [23]. The *classical real or complex Laguerre plane* is the miquelian Laguerre plane over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ (with suitable topologies) and can be obtained as the geometry of non-trivial plane sections of an elliptic cone in 3-dimensional projective space over $\mathbb{F}$ with its vertex removed. By replacing the elliptic cone in the construction of the classical real Laguerre plane by a cone over an arbitrary oval, i.e., a convex, differentiable simply closed curve in $\mathbb{R}^2$, we also obtain 2-dimensional Laguerre planes. These are the so-called *2-dimensional ovoidal Laguerre planes*. There is no 4-dimensional analogue because every closed oval in the Desarguesian complex projective plane is a conic; compare [18], 55.13.

An *automorphism* of a Laguerre plane is a permutation of the point set such that parallel classes are mapped to parallel classes and circles are mapped to circles. The *kernel* T of a Laguerre plane consists of all automorphisms that fix each parallel class. This collection of automorphisms is a normal subgroup of the group of all automorphisms. The collection of all continuous automorphisms of a $2n$-dimensional Laguerre plane $\mathcal{L}$ is a Lie group with respect to the compact-open topology, the automorphism group Γ of $\mathcal{L}$, see [4], Satz 3.9, or [21], and T is a closed normal subgroup of Γ.

A Laguerre plane is an *elation Laguerre plane* if there exists a group Δ of automorphisms in the kernel T that acts regularly on the set of circles. The 2-dimensional elation Laguerre planes are precisely the 2-dimensional ovoidal Laguerre planes. A 4-dimensional Laguerre plane $\mathcal{L}$ is an elation Laguerre plane if and only if the collection of all automorphisms in T that fix no circle, plus the identity, which is a closed normal subgroup of T, acts transitively on the set of circles, see [22]. $2n$-dimensional elation Laguerre planes can be characterized in terms of transitivity properties or the dimension of T; see [22], [24], Theorem 2.7, and [16], section 5.4.2.

The stabilizer Δ_p of a point p induces elations of the *derived projective plane $\mathcal{P}_p$ at p*, that is, the projective completion of the *derived affine plane $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ at p* whose point set $A_p \approx \mathbb{R}^{2n}$ consists of all points of $\mathcal{L}$ that are not parallel to p and whose line set $\mathcal{L}_p$ consists of all restrictions to A_p of circles of $\mathcal{L}$ passing through p and of all parallel classes not passing through p. All these elations have the common centre ω, the point at infinity of lines that come from parallel classes of the Laguerre plane. In fact, each derived projective plane of a $2n$-dimensional Laguerre plane is a topological locally compact connected $2n$-dimensional projective plane and, in case of an elation Laguerre plane, even a dual translation plane with centre ω.

In an elation Laguerre plane there is for each parallel class G a subgroup Δ_G of the elation group Δ that acts trivially on G and regularly on each other parallel class. These kinds of groups, among others, were used by Kleinewillinghöfer's in her classification of Laguerre planes, see the following section.

3 Kleinewillinghöfer types of 4-dimensional elation Laguerre planes

Similar to the Lenz–Barlotti classification of projective planes Kleinewillinghöfer classified Laguerre planes with respect to groups of *central automorphisms*, that is, automorphisms such that at least one point is fixed and central collineations are induced in the derived projective plane at one of the fixed points. More precisely, one considers linearly transitive groups of central automorphisms, that is, the induced groups of central collineations are transitive on each central line except for the obvious fixed points, the centre and the point of intersection with the axis. Note that, because each collineation induced by an automorphism of the Laguerre plane fixes the line W at infinity, one has that the centre of the collineation is on W or the axis equals W. We say that a Laguerre plane $\mathcal{L}$ is of Kleinewillinghöfer type X if the full automorphism group of $\mathcal{L}$ is of type X, see [9], [10] or [16] for the definitions of the various Kleinewillinghöfer types.

One particular kind of central automorphisms of a Laguerre plane $\mathcal{L}$ are *Laguerre translations*. These are automorphisms of $\mathcal{L}$ that fix a point p and all points on the parallel class $|p|$ of p and induce translations in the derived affine plane at p. If, moreover, each parallel class is fixed, one speaks of a $|p|$-*translation*. (There are other kinds of Laguerre translations, those fixing each circle in a touching pencil $B(p, C) = \{C' \in \mathcal{C} \mid C' \text{ touches } C \text{ at } p\}$ with support p for some point p and circle $C \ni p$.) In each derived projective plane at one of the fixed points we have an elation with axis the line at infinity.

With respect to Laguerre translations Kleinewillinghöfer obtained 11 types, labeled A to K; see [9] Satz 3.3, or [10] Satz 2. In case of elation Laguerre planes the set of all parallel classes G for which the group of all G-translations is linearly transitive equals the set Π of all parallel classes of $\mathcal{L}$. Hence only types D, J and K are possible. More precisely, let $\mathcal{E}$ denote the set of all parallel classes for which the automorphism group Γ of the Laguerre plane is linearly transitive and let $\mathcal{B}$ denote the set of all touching pencils $B(p, C)$ for which the automorphism group is linearly transitive. Then in type D one has $|\mathcal{E}| \geq 3$ and $\mathcal{B} = \emptyset$, in type J one has $\mathcal{E} = \Pi$ and there is a parallel class G such that $\mathcal{B}$ consists of all touching pencils with support on G, and in type K one has $\mathcal{E} = \Pi$ and $\mathcal{B}$ consists of all touching pencils.

In types J and K there are linearly transitive groups of Laguerre translations that induce elations in some derived projective plane with centre different from ω, the point at infinity of vertical lines (parallel classes). Hence such a derived plane is a translation plane and also a dual translation plane.

Proposition 3.1 *There is no 4-dimensional elation Laguerre plane of Kleinewillinghöfer type J and a 4-dimensional elation Laguerre plane of type K is classical.*

Proof. Let $\mathcal{L}$ be a 4-dimensional elation Laguerre plane such that there is a touching pencil $B(p, C)$ with respect to which the group of Laguerre translations of $\mathcal{L}$ is linearly transitive. The derived projective plane at p then is both a 4-dimensional translation plane and a 4-dimensional dual translation plane. Hence this derived plane must be Desarguesian by [18], Theorem 72.11. But then $\mathcal{L}$ is the classical complex Laguerre plane by [12], Corollary 2.5. Indeed, the classical complex Laguerre plane is of Kleinewillinghöfer type VII.K.13 and thus type J is not possible in 4-dimensional elation Laguerre planes. □

The description of type D as given by Kleinewillinghöfer does not directly relate to elation Laguerre planes because $|\mathcal{E}| \geq 3$ may not imply $\mathcal{E} = \Pi$. However, in the 4-dimensional case, we can conclude that we must have an elation Laguerre plane, see also [25], Corollary 3.5, where the same result is implicitly obtained although along a different method.

Proposition 3.2 *If the set $\mathcal{E}$ of all parallel classes for which the 4-dimensional Laguerre plane $\mathcal{L}$ is linearly transitive contains at least 3 parallel classes, then $\mathcal{L}$ is is an elation Laguerre plane.*

Proof. Let G_1, G_2, G_3 be three distinct parallel classes in $\mathcal{E}$. The G_i-translations form a normal subgroup Δ_i in the kernel T of $\mathcal{L}$. Since by assumption Δ_i is linearly transitive, Δ_i is 2-dimensional. Furthermore, $\Delta_i \cap \Delta_j = \{id\}$ for $i \neq j$, so that Δ_i and Δ_j generate a normal 4-dimensional subgroup Δ_{ij} of T.

Assume that there are $\gamma_i \in \Delta_i$, $i = 1, 2, 3$, and a circle C of $\mathcal{L}$ such that $\gamma_1\gamma_2\gamma_3$ fixes C. Consider the circles $C_1 = \gamma_1^{-1}(C)$ and $C_3 = \gamma_3(C)$. Then C_i touches C in $p_i = C \cap G_i$ for $i = 1, 3$ and, because $C_1 = \gamma_2(C_3)$, the circle C_1 also touches C_3 in $p_2 = C_3 \cap G_2$. We consider the Lie geometry $\mathcal{Q}$ associated with $\mathcal{L}$. This geometry has points the points of $\mathcal{L}$ plus the circles of $\mathcal{L}$ plus one additional point at infinity ∞; the lines of $\mathcal{Q}$ are the extended parallel classes $G \cup \{\infty\}$ for $G \in \Pi$ and the extended tangent pencils $B(p, C) \cup \{p\}$ with incidence being the natural one (compare [20], Chapter 3, or [19]). The circles C, C_1, C_3 give rise to three points in $\mathcal{Q}$, any two of which are collinear. But $\mathcal{Q}$ is a generalized quadrangle by [20], Theorem 3.4, so that no proper triangles exist in $\mathcal{Q}$. Therefore two of the points (i.e., circles in $\mathcal{L}$) must be the same which then implies $\gamma_1 = \gamma_2 = \gamma_3 = id$.

This shows that $\Delta_3 \cap \Delta_{12} = \{id\}$. Hence Δ_1, Δ_2, Δ_3 generate a normal 6-dimensional subgroup Δ_{123} of T. But Δ_{123} consists of elations of $\mathcal{L}$ so that the elation group of $\mathcal{L}$ is 6-dimensional. Thus $\mathcal{L}$ is an elation Laguerre plane by [22], Proposition 4.1. □

The following two Corollaries are immediate consequences of Propositions 3.1 and 3.2 because Laguerre planes of Kleinewillinghöfer types J and K contain three parallel classes as in Proposition 3.2.

Corollary 3.3 *There is no 4-dimensional Laguerre plane of Kleinewillinghöfer type J and a 4-dimensional Laguerre plane of type K is classical.*

Note the subtle but important difference between the above Corollary and Proposition 3.1. The proposition is about 4-dimensional elation Laguerre planes whereas Corollary 3.3 deals with 4-dimensional Laguerre planes that are not necessarily assumed to be elation Laguerre planes.

Corollary 3.4 *A non-classical 4-dimensional Laguerre plane is an elation Laguerre plane if and only if it is of Kleinewillinghöfer type D.*

The first part of Corollary 3.3 can be strengthened to exclude Kleinewillinghöfer type I as well thus providing an alternative proof for the exclusion of type J.

Proposition 3.5 *There is no 4-dimensional Laguerre plane of Kleinewillinghöfer type I or J.*

Proof. In type I there is a parallel class G such that $\mathcal{E} = \{G\}$ and $\mathcal{B}$ consists of all touching pencils with support on G. Let p and q be two distinct points on G. The derived projective planes $\mathcal{P}_p$ and $\mathcal{P}_q$ at p and q, respectively, are 4-dimensional translation planes. Furthermore, each of the translations of $\mathcal{P}_p$ and $\mathcal{P}_q$ is induced by a Laguerre translation. Let Σ_p and Σ_q be the group of all Laguerre translations that induces the translation group of $\mathcal{P}_p$ and $\mathcal{P}_q$, respectively. An automorphism $\alpha \in \Sigma_p \cap \Sigma_q$ but not in T fixes a circle through p and a circle through q and thus the intersection of these two circles. However in a 4-dimensional Laguerre plane any two distinct circles have at least one point in common (and at most two such points). This implies that α fixes a point and thus must be the identity in contradiction to $\alpha \notin T$. Hence $\Sigma_p \cap \Sigma_q \leq T$ and the two groups are distinct. Since Σ_q acts trivially on G by the definition of Laguerre translations, we see that Σ_q induces a 4-dimensional abelian group of collineations of $\mathcal{P}_p$. Hence $\mathcal{P}_p$ admits two distinct 4-dimensional abelian groups of collineations. By [18], Theorem 72.12, therefore $\mathcal{P}_p$ is Desarguesian. But then $\mathcal{L}$ is classical and thus of type K. This shows that type I is not possible in 4-dimensional Laguerre planes.

Since type J contains a configuration as in type I, Kleinewillinghöfer type J is not possible either. □

Kleinewillinghöfer further investigated two other kinds of central Laguerre automorphisms, Laguerre homologies and Laguerre homotheties. A *Laguerre homology* of a Laguerre plane $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points of a circle. One then speaks of a *C-homology* if C is the circle that is fixed. In each derived projective plane at one of the fixed points we have a homology with axis the line induced by C and centre the point ω. In particular, each Laguerre homology belongs to the kernel of $\mathcal{L}$. With respect to Laguerre homologies Kleinewillinghöfer obtained seven types, labeled I to VII. Since the elation group of an elation Laguerre plane is transitive on the set of circles $\mathcal{C}$, the set of all circles C for which the group of all C-homologies is linearly transitive is either empty or all of $\mathcal{C}$ in case of an elation Laguerre plane. Hence only types I and VII are possible in elation Laguerre planes.

In Kleinewillinghöfer type VII each derived projective plane $\mathcal{P}_p$ is (ω, L)-transitive for each line $L \not\ni \omega$ of $\mathcal{P}_p$ and hence is Desarguesian by [15], 3.2.27. For a 4-dimensional elation Laguerre plane we thus obtain, as before, the classical complex Laguerre plane.

Proposition 3.6 *A 4-dimensional Laguerre plane of type VII is classical and a non-classical 4-dimensional elation Laguerre plane is of type I.*

Finally, a *Laguerre homothety* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that fixes two non-parallel points and induces a homothety in the derived affine plane at each of these two fixed points. One then speaks of a $\{p, q\}$*-homothety* if p, q are the two fixed points. In each derived projective plane at one of the fixed points we have a homology with axis the line at infinity and centre the other fixed point (a homothety of the derived affine plane). With respect to Laguerre homotheties Kleinewillinghöfer obtained 13 types, labeled 1 to 13.

The transitivity of the elation group on the set of circles implies that the set of all unordered pairs of nonparallel points $\{p, q\}$ for which the group of all $\{p, q\}$-homotheties is linearly transitive is either empty or contains all unordered pairs $\{x, y\}$ with $x \in G$, $y \in H$ for two distinct parallel classes G and H (e.g., $G = |p|$ and $H = |q|$) in case of an elation Laguerre plane. Hence only types 1, 8, 12 or 13 are possible in elation Laguerre planes. More precisely, let $\mathcal{H}$ denote the set of all unordered pairs of nonparallel points $\{p, q\}$ for which the group of all $\{p, q\}$-homotheties is linearly transitive. Then in types 1, 8, 12 and 13 one has $\mathcal{H} = \emptyset$, $\mathcal{H} = \{\{p, q\} \mid p \in G, q \in H\}$ for two distinct parallel classes G and H, $\mathcal{H} = \{\{p, q\} \mid p \in G, q \notin G\}$ for a parallel classes G, and $\mathcal{H}$ all pairs of nonparallel points, respectively. Type 13 characterizes the miquelian Laguerre planes, see [7], Satz 7.

Proposition 3.7 *There is no 4-dimensional elation Laguerre plane of Kleinewillinghöfer type 8 or 12 and a 4-dimensional Laguerre plane of type 13 is classical.*

Proof. In a 4-dimensional elation Laguerre plane $\mathcal{L}$ that contains an unordered pair of nonparallel points $\{p, q\}$ for which the automorphism group of $\mathcal{L}$ is linearly transitive, the derived projective plane at p is a dual translation plane that admits a 2-dimensional group of homologies with an affine centre. The dual of [18], Theorem 72.11, then implies that the derived plane is Desarguesian so that $\mathcal{L}$ is again classical. □

Since, as seen above, only types I, D and 1 are possible for non-classical 4-dimensional elation Laguerre planes with respect to Laguerre homologies, Laguerre translations and Laguerre homotheties, respectively, we finally obtain the following.

Theorem 3.8 *A non-classical 4-dimensional Laguerre plane is an elation Laguerre plane if and only if it is of Kleinewillinghöfer type I.D.1. The 4-dimensional elation Laguerre planes are precisely the 4-dimensional Laguerre planes of Kleinewillinghöfer types I.D.1 and VII.K.13.*

4 A look at other Laguerre planes

In [17] the possible Kleinewillinghöfer types of 2-dimensional Laguerre planes where determined. Since the 2-dimensional elation Laguerre planes are precisely the ovoidal Laguerre planes, [17], Lemma 3.1, characterizes the 2-dimensional elation Laguerre planes as those 2-dimensional Laguerre planes of Kleinewillinghöfer type VII and Corollary 4.5 characterizes the non-classical ones as those 2-dimensional Laguerre planes of Kleinewillinghöfer type D. There are no 2-dimensional Laguerre planes of type 12; see [17], Corollary 5.3. However, there are non-classical 2-dimensional elation Laguerre planes of combined Kleinewillinghöfer types VII.D.1 and VII.D.8. This shows that Theorem 3.8 does not generalize to a characterization of elation Laguerre planes among arbitrary Laguerre planes.

Combining the results of [17] and Corollary 3.4 we obtain the following.

Theorem 4.1 *A non-classical $2n$-dimensional Laguerre plane, $n = 1, 2$, is an elation Laguerre plane if and only if it is of Kleinewillinghöfer type D.*

Although Theorem 4.1 is still valid for finite Laguerre planes of odd order, see the remarks below, it seems hard to extend it beyond the class of 2- or 4-dimensional Laguerre planes or finite Laguerre planes of odd order. To do so and following the same route for such a class of Laguerre planes as in the 4-dimensional case pursued here requires to overcome three problems. Firstly, find an analogue of Proposition 3.2. In its proof we used that the Lie geometry of the Laguerre plane is a generalized quadrangle which is not true for every Laguerre plane. Using order arguments, however, this was avoided in the 2-dimensional case but could have been done in exactly the same way as in the proof of Proposition 3.2 (halving all the dimensions being referred to in the proof). Secondly, we used repeatedly that translation plane and dual translation plane implies Desarguesian. This of course is not true in general. There are many examples of semifields that are not skew fields; these then coordinatise non-Desarguesian planes that are both translation and dual translation planes, see for example [3] Section 5.3. Thirdly, we used several times that a Desarguesian derivation implies that the Laguerre plane is miquelian. Again this is not true in general as ovoidal Laguerre planes show but one can get around this in the 2-dimensional case.

For finite Laguerre planes of odd order, the first and third properties can be verified, compare [14] VII.1 and VII.2 or [2], but certainly the second property need not be true. (Note however that no non-miquelian finite Laguerre planes of odd order are known but it seems that elation Laguerre planes are the best candidates so far to find some.) So the proof of Proposition 3.2 goes through. In order to still obtain the analogous result of Corollary 3.3 for these finite planes Hartmann [7] used functional identities for circle describing functions; see [7], Satz 3. But then Corollary 3.4 also follows (where 4-dimensional is replaced by finite of odd order).

Theorem 4.2 *A non-miquelian finite Laguerre plane of odd order is an elation Laguerre plane if and only if it is of Kleinewillinghöfer type D.*

References

[1] O. Bröcker. Eight and seven point degenerations of Miquel's Theorem in Laguerre planes *J. Geom.*, **75** (2002), 46–60.

[2] Y. Chen and G. Kaerlein. Eine Bemerkung über endliche Laguerre- und Minkowski-Ebenen. *Geom. Dedicata* **2** (1973), 193–194.

[3] P. Dembowski *Finite Geometries.* Springer, Berlin, Heidelberg, 1968.

[4] A. Förtsch. *Kollineationsgruppen endlichdimensionaler topologischer Benzebenen.* Dissertation, Erlangen, 1982.

[5] H. Groh. Topologische Laguerreebenen I. *Abh. Math. Sem. Univ. Hamburg*, **32** (1968), 216–231.

[6] H. Groh. Topologische Laguerreebenen II. *Abh. Math. Sem. Univ. Hamburg*, **34** (1970), 11–21.

[7] E. Hartmann Transitivitätssätze für Laguerre-Ebenen. *J. Geom.*, **18** (1982), 9–27.

[8] M. Joswig. Pseudo-ovals, elation Laguerre planes, and translation generalized quadrangles *Beitr. Algebra Geom.*, **40** (1999), 141–152.

[9] R. Kleinewillinghöfer. *Eine Klassifikation der Laguerre-Ebenen.* Dissertation, TH Darmstadt, 1979.

[10] R. Kleinewillinghöfer. Eine Klassifikation der Laguerre-Ebenen nach $\mathcal{L}$-Streckungen und $\mathcal{L}$-Translationen. *Arch. Math.*, **34** (1980), 469–480.

[11] N. Knarr. A geometric characterization of elation Laguerre planes. *Arch. Math.*, **78** (2002), 162–165.

[12] R. Löwen. *Projectivities and the geometric structure of topological planes.* Geometry - von Staudt's point of view (eds P. Plaumann and K. Strambach), Bad Windsheim, 1981, 339–372.

[13] R. Löwen. Topological pseudo-ovals, elation Laguerre planes, and elation generalized quadrangles. *Math. Z.*, **216** (1994), 347–369.

[14] S.E. Payne and J.A. Thas. Generalized quadrangles with symmetry. Part II. *Simon Stevin* **49** (1976), 81–103.

[15] G. Pickert. *Projektive Ebenen.* 2nd ed. Springer, Berlin, Grundlehren der Math. Wissenschaften vol. 80, 1975.

[16] B. Polster and G.F. Steinke. *Geometries on Surfaces.* Cambridge University Press, Encyclopedia of Mathematics and its Applications vol 84, 2001.

[17] B. Polster and G.F. Steinke. On the Kleinewillinghöfer types of flat Laguerre planes. *Result. Math.*, **46** (2004), 103–122.

[18] H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen and M. Stroppel. *Compact Projective Planes*. de Gruyter, Berlin, 1995.

[19] A.E. Schroth Generalized quadrangles constructed from topological Laguerre planes. *Geom. Dedicata*, **46** (1993), 339–361.

[20] A.E. Schroth. *Topological circle planes and topological quadrangles.* Pitman Research Notes in Mathematics Series, vol. 337, Longman Essex, UK, 1995.

[21] G.F. Steinke. The automorphism group of Laguerre planes. *Geom. Dedicata*, **21** (1986), 55–58.

[22] G.F. Steinke. The elation group of a 4-dimensional Laguerre plane. *Monatsh. Math.*, **111** (1991), 207–231.

[23] G.F. Steinke. Topological circle geometries. Handbook of incidence geometry (Ed. F. Buekenhout), pp. 1325–1354, Elsevier, Amsterdam, 1995.

[24] G.F. Steinke. On 4-dimensional elation Laguerre planes admitting simple Lie groups of automorphisms. *Forum Math.*, **11** (1999), 79–103,

[25] G.F. Steinke and H. Van Maldeghem. Generalized Quadrangles and Projective Axes of Symmetry. *Beitr. Algebra Geom.*, **51** (2010), 191–207.