Variations on a theme by Ishihara

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Abstract

Ishihara's tricks have been a useful tool in constructive mathematics whenever one has to deal with strongly extensional mappings. In this short note, we try to weaken some of their assumptions and apply them a more general case.

1 Preliminaries

We assume that the reader is familiar with Bishop style constructive mathematics (BISH) and the constructive theory of metric space [2], and only remind him that a mapping $f : X \to Y$ between two metric spaces (X, ρ_X) and (Y, ρ_Y) is called *strongly extensional* if for all $y, y' \in Y$

$$\rho_Y(f(y), f(y')) > 0 \implies \rho_X(y, y') > 0 \; .$$

In all the major varieties (i.e. informal models) of BISH all functions defined on a complete metric space are strongly extensional.

Ishihara's tricks [6] enable one to make a—from a constructive viewpoint—rather surprising decision. For completeness' sake and for future reference we are going to state them here.

Proposition 1.1 (Ishihara's first trick). *Let f be a strongly extensional mapping of a complete metric space X into a metric space Y, and let $(x_n)_{n\geqslant 1}$ be a sequence in X converging to a limit x. Then for all positive numbers $\alpha < \beta$, either $\rho(f(x_n), f(x)) > \alpha$ for some n or $\rho(f(x_n), f(x)) < \beta$ for all n.*

Proposition 1.2 (Ishihara's second trick). *Let f be a strongly extensional mapping of a complete metric space X into a metric space Y, and let $(x_n)_{n\geqslant 1}$ be a sequence in X converging to a limit x. Then for all positive numbers $\alpha < \beta$, either $\rho(f(x_n), f(x)) < \beta$ eventually or $\rho(f(x_n), f(x)) > \alpha$ infinitely often.*

We would like to start by proving that in case the second alternative holds in Ishihara's second trick also the limited principle of omniscience (**LPO**) holds.[1]

[1] This has already been proven in [3], but is included here since details of the proof are of importance later in the paper.

The limited principle of omniscience states that for every binary sequence $(a_n)_{n\geqslant 1}$ we can decide whether

$$\forall n \in \mathbb{N}\ (a_n = 0) \vee \exists n \in \mathbb{N}\ (a_n = 1)\ .$$

LPO is provably wrong in most constructive forms of mathematics. We will, later in the paper, also be interested in a slightly weaker form: the weak limited principle of omniscience (**WLPO**), that states that for every binary sequence $(a_n)_{n\geqslant 1}$ we can decide whether

$$\forall n \in \mathbb{N}\ (a_n = 0) \vee \neg\forall n \in \mathbb{N}\ (a_n = 0)\ .$$

After these definitions we can now give the proof linking Ishihara's second trick to **LPO**. A reader not familiar with the proof of Ishihara's tricks will be able to get an idea of the techniques used.

Proof. Assume that $\rho\,(f(x_n), f(x)) > \alpha > 0$ infinitely often, i.e. there exists a sequence $y_n \to x$ such that $\rho(f(y_n), f(x)) > \alpha > 0$ for all $n \in \mathbb{N}$. W.l.o.g. we may also assume that $\rho(y_n, x) < 2^{-n}$ for all $n \in \mathbb{N}$. Now consider an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$. Define a sequence $(z_n)_{n\geqslant 1}$ in X by

$$z_n = \begin{cases} y_m & \text{if } \lambda_n = 1 \text{ and } \lambda_m = 1 - \lambda_{m+1}, \\ x & \text{if } \lambda_n = 0. \end{cases} \tag{1}$$

Then z_n is a Cauchy sequence, since $\rho(z_i, z_j) < 2^{-(m-1)}$ for $i, j > m$. It thus converges to a limit $z \in X$. Now either $\rho(f(x), f(z)) < \alpha$, in which case $\lambda_n = 0$ for all $n \in \mathbb{N}$ or $\rho(f(x), f(z)) > \alpha/2$. In the second case, since f is strongly extensional $\rho(x, z) > 0$, which means that there exists $N \in \mathbb{N}$ such that $\rho(x, z) > 2^N$. For this N, if $\lambda_N = 0$, we would get the contradiction

$$2^N \geqslant \rho(z_N, z) = \rho(x, z) > 2^N.$$

Hence $\lambda_N = 1$. □

As we can see in this proof, we do not need full completeness of the space, but only that Cauchy sequences of a very special form, such as in (1), have a limit. We will formalise this idea in the next section, and also show, by means of an example, that we really enlarge the class of spaces that Ishihara's tricks apply to. In the section thereafter, we remove the assumption of strong extensionality.

2 Weakening completeness

Let (X, ρ) be a metric space. For a sequence $x = (x_n)_{n\geqslant 1}$ in X converging to $x_\infty \in X$ and an increasing binary sequence $\lambda = (\lambda_n)_{n\geqslant 1}$ we define a sequence $\lambda \circledast x$ by

$$(\lambda \circledast x)_n = \begin{cases} x_m & \text{if } \lambda_n = 1 \text{ and } \lambda_m = 1 - \lambda_{m+1}, \\ x_\infty & \text{if } \lambda_n = 0. \end{cases}$$

As the next Lemma shows this construction always yields a Cauchy sequence.

Lemma 2.1. *If $x = (x_n)_{n \geqslant 1}$ is a sequence in X converging to $x_\infty \in X$ and $\lambda = (\lambda_n)_{n \geqslant 1}$ an increasing binary sequence then $\lambda \circledast x$ is a Cauchy sequence.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since $(x_n)_{n \geqslant 1}$ converges to x_∞ there exists $N \in \mathbb{N}$ such that $\rho(x_n, x_\infty) < \frac{\varepsilon}{2}$ for all $n \geqslant N$. If $\lambda_N = 1$, then $(\lambda \circledast x)_n = x_M$ for all $n \geqslant N$, where $M \leqslant N$ is the smallest number such that $\lambda_M = 1$. Therefore in this case

$$\rho\left((\lambda \circledast x)_i, (\lambda \circledast x)_j\right) = \rho\left(x_M, x_M\right) < \varepsilon \ ,$$

for all $i, j \geqslant N$.

If $\lambda_N = 0$ and $i \geqslant N$, then, similar to the above, either $(\lambda \circledast x)_i = x_M$, but this time for some $M > N$, or $(\lambda \circledast x)_i = x_\infty$. In both cases $|(\lambda \circledast x)_i - x_\infty| < \frac{\varepsilon}{2}$, and therefore

$$|(\lambda \circledast x)_i - (\lambda \circledast x)_j| < \varepsilon$$

for all $i, j \geqslant N$. □

We will call a metric space (X, ρ) **complete enough**, if for every sequence $x = (x_n)_{n \geqslant 1}$ in X converging to $x_\infty \in X$ and every increasing binary sequence $\lambda = (\lambda_n)_{n \geqslant 1}$ the (Cauchy) sequence $\lambda \circledast x$ converges in X. Under the assumption of **LPO** every metric space is complete enough. The Lemma above shows that completeness of a space implies that the space is complete enough. The reverse is not true (even classically), as the following example shows.

Example 2.2. *The space* $\mathbf{P}$ *of all permutations* $\pi : \mathbb{N} \to \mathbb{N}$ *is a complete enough, separable subspace of Baire space. It is, however, not complete.*

Proof. To see that $\mathbf{P}$ is complete enough, consider a sequence $(\sigma_n)_{n \geqslant 1}$ in $\mathbf{P}$ that converges to a sequence $\sigma \in \mathbb{N}^{\mathbb{N}}$, and let $(\lambda_n)_{n \geqslant 1}$ be an increasing binary sequence. Furthermore let μ be the limit of $\lambda \circledast \sigma$ in $\mathbb{N}^{\mathbb{N}}$. We will show surjectiveness and injectivity of μ simultaneously. To this end let $k, i, j \in \mathbb{N}$ be arbitrary. Set $m = \sigma^{-1}(k)$ and $K = \max\{m, i, j\}$. Since σ_n converges to σ, there exists N such that

$$\forall n \geqslant N \; (\overline{\sigma} K = \overline{\sigma_n} K).$$

If $\lambda_N = 0$, then $\overline{\mu} K = \overline{\sigma}(K)$. Hence $\mu(m) = \sigma(m) = k$ and $\mu(i) = \sigma(i) \neq \sigma(j) = \mu(j)$. If $\lambda_n = 1$, then $\mu = \sigma_M$ for the biggest M such that $\lambda_M = 1$. Thus in both cases there exists m' such that $\mu(m') = k$ and $\mu(i) \neq \mu(j)$, which means that μ is a permutation.

To prove separability we first note that the set

$$A_n = \{\alpha \in \mathbb{N}^n \mid \forall 1 \leqslant i, j \leqslant n \ (i \neq j \implies \alpha(i) \neq \alpha(j))\}$$

is countable for every $n \in \mathbb{N}$. For $\alpha \in A_n$, we define recursively

$$\widetilde{\alpha}(k) = \begin{cases} \alpha(k) & \text{if } k \leqslant n, \\ \min\{i \mid i \notin \{\widetilde{\alpha}(1), \ldots, \widetilde{\alpha}(k-1)\} & \text{if } n < k \leqslant \max\{\alpha(1), \ldots, \alpha(n)\}, \\ k & \text{otherwise.} \end{cases}$$

Then $\widetilde{\alpha}$ is a permuation, such that $\overline{\widetilde{\alpha}}n = \alpha$. Let $\widetilde{A}$ be the countable set of all these permutations. For an arbitrary $\sigma \in \mathbf{P}$ and $n \in \mathbb{N}$, the permutation $\widetilde{\overline{\sigma}N} \in \widetilde{A}$ is such that

$$\rho(\widetilde{\overline{\sigma}N}, \sigma) \leqslant 2^{-N},$$

where ρ is the usual metric on Baire space. Therefore $\mathbf{P}$ is separable.

To see that $\mathbf{P}$ is not complete consider the sequence of permutations $(\sigma_n)_{n\geqslant 1}$ defined by

$$\sigma_n(k) = \begin{cases} k+1 & \text{for } k < n, \\ 0 & \text{for } k = n, \\ k & \text{for } k > n. \end{cases}$$

Then $\sigma_n \to S$, where S is the successor function, which is not a permutation. (Notabene: The limit of permutations is, however, always injective). $\square$

By inspection of the proofs of Ishihara's tricks in [4] we can weaken completeness to complete enough and get:

Lemma 2.3. *Let f be a strongly extensional mapping of a complete enough metric space into a metric space Y, and let $(x_n)_{n\geqslant 1}$ be a sequence in X converging to a limit x. Then for all $\epsilon > 0$ either $\rho(f(x_n), f(x)) < \epsilon$ eventually, or $\rho(f(x_n), f(x)) > \epsilon/2$ infinitely often. In case the second alternative holds,* ***LPO*** *holds.*

3 Removing strong extensionality

In this section we investigate whether it is possible to drop the strong extensionality assumption. This is indeed possible and Ishihara's first trick remains almost unchanged.

Proposition 3.1. *Let f be a mapping of a complete metric space X into a metric space Y, and let $(x_n)_{n\geqslant 1}$ be a sequence in X converging to a limit x. Then for all positive numbers $\alpha < \beta$ and all $\epsilon > 0$, either $\rho(f(x_n), f(x)) < \beta$ for all n or there exists $y \in X$ such that $\rho(x, y) < \epsilon$ and $\rho(f(x), f(y)) > \alpha$.*

Proof. Like the normal proof, but with the sequence replaced by its tail that is within the ϵ neighbourhood of x. $\square$

The result of the second trick is changed considerably without strong extensionality. The next proposition could be, wrongly, but memorably, be rephrased as "Either a function on a complete metric space is sequentially non-discontinuous or **WLPO** holds."

Proposition 3.2. *Let f be a mapping of a complete metric space X into a metric space Y, and let $(x_n)_{n\geqslant 1}$ be a sequence in X converging to a limit x. Then for all positive numbers $\alpha < \beta$, either it is impossible that $\rho(f(x_n), f(x)) \geqslant \beta$ infinitely often or there exists a sequence $(y_n)_{n\geqslant 1}$ in X that converges to x and is such that $\rho(f(x), f(y_n)) > \alpha$. Furthermore in the latter case* ***WLPO*** *holds.*

Proof. Using the previous proposition fix a binary increasing sequence $(\lambda_n)_{n\geqslant 0}$ such that

$$\begin{aligned}\lambda_n = 0 &\implies \exists y_n \in B_{2^{-n}}(x)\,(\rho(f(x), f(y_n)) > \alpha)\\ \lambda_n = 1 &\implies \forall i \geqslant n\,(\rho(f(x), f(x_i)) < \beta)\end{aligned}$$

We may assume that $\lambda_0 = 0$. Define a sequence $(z_n)_{n\geqslant 1}$ by

$$z_n = \begin{cases} y_{k-1} \text{ when } \lambda_n = 1 \text{ and } k = \min_{i\leq n}\{\lambda_k = 1\},\\ x \text{ whenever } \lambda_n = 0.\end{cases}$$

Then $(z_n)_{n\geqslant 1}$ is a Cauchy sequence that converges to a limit z. Now either $\rho(f(z), f(x)) < \alpha$ or $\rho(f(x), f(z)) > 0$. In the second case it is impossible that infinitely many terms of (λ_n) are zero, since then $z = x$. In the first case there cannot be a n such that $\lambda_n = 1$. □

Corollary 3.3. *If* $\neg$***WLPO*** *holds then every mapping on a complete enough metric space is sequentially non-discontinuous.*

As an immediate consequence we have re-proven Markov's 1954 result that (in Russian recursive mathematics) every mapping is (sequentially) non-discontinuous.

We also get the curious

Corollary 3.4.

1. *If a function f on a complete enough metric space is such that*

   $$\boldsymbol{WLPO} \implies f \textit{ is (sequentially) non-discontinuous},$$

   then f is already (sequentially) non-discontinuous.

2. *If in addition f is also strongly extensional and*

   $$\boldsymbol{LPO} \implies f \textit{ is sequentially continuous},$$

   then f is already sequentially continuous.

We would like to stress that these results hold for an arbitrary function in BISH and without any additional assumptions.

4 Applications

We will show the usefulness of the results by giving three applications.

4.1 Injections from Baire space into the natural numbers

In his blog[2] A. Bauer relates a question of F. Richman whether or not there is a purely constructive proof that there is no injection from Baire space ($\mathbb{N}^{\mathbb{N}}$)

[2] http://math.andrej.com/2011/06/15/constructive-gem-an-injection-from-baire-space-to-natural-numbers/

into the natural numbers. Earlier work by P. Oliva and M. Escardó had shown that there are computer programs witnessing the fact that there are no such injections. This, however, was not quite enough to answer the questions, since there are subtle differences between recursive models and pure Bishop-style constructive mathematics. And indeed, A. Bauer was able to show that—rather surprisingly—there is a model (based on infinite time Turing machines), in which there is an injection from Baire space into the natural numbers. Even though a positive answer is therefore impossible, using our extension of Ishihara's tricks we can, nevertheless, give interesting partial results.

Proposition 4.1. *If $H : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a strongly extensional function, then either there exist $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ with $\alpha \neq \beta$ and $H(\alpha) = H(\beta)$ or **LPO** holds.*

Proof. Let δ_k be the binary sequence having exactly one 1 at its kth place and is zero everywhere else. Clearly $\delta_k \to 0$, where 0 is the constant zero function, in $\mathbb{N}^{\mathbb{N}}$. Using Ishihara's second trick either there is n such that $H(\lambda_n) = H(0)$, or **LPO** holds. Since $\delta_n \neq 0$ we are done. ☐

Even without the assumption of strongly extensionality we get the interesting

Proposition 4.2. *If $H : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a function, then either there exist $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ with $\neg(\alpha = \beta)$ and $H(\alpha) = H(\beta)$ or **WLPO** holds.*

Proof. Let $(\delta_k)_{k\geqslant 1}$ as in the previous proof. By Proposition 3.2 either **WLPO** holds, or it is impossible that $H(\delta_k) \neq H(0)$ infinitely often. Now fix a binary sequence $(\lambda_k)_{k\geqslant 1}$ such that

$$\begin{aligned} \lambda_k = 0 &\implies H(\delta_k) \neq H(0) \\ \lambda_k = 1 &\implies H(\delta_k) = H(0)\ . \end{aligned}$$

Since $\mathbb{N}^{\mathbb{N}}$ is complete it is complete enough, and therefore the sequence $\lambda \circledast \delta$ has a limit α. Assume that $H(\alpha) \neq H(0)$. Then there cannot be a k with $\lambda_k = 1$, since in that case $\alpha = \lambda_{k'}$, where k' is the smallest natural number smaller than or equal to k such that $\lambda_{k'} = 1$. Hence $\lambda_k = 0$ for all $k \in \mathbb{N}$, which means that $H(\delta_k) \neq H(0)$ infinitely often; a contradiction. Thus $H(\alpha) = H(0)$ and we are done. ☐

One would maybe hope that one could use the fairly strong **LPO** or **WLPO** to deduce, in case the second alternative holds, that the function is injective in this case as well. However, A. Bauer's model shows that **LPO** is not enough for this.

4.2 Riemann's permutation theorem

Riemann's permutation theorem states that if a series of reals is conditionally convergent, that is it is convergent but not absolutely convergent, then it can be rearranged to converge to any real number or even diverge to infinity. Berger and Bridges have shown that this can be proven constructively [1]. However, the classically equivalent statement that if every rearrangement converges, then the series is absolutely convergent, is surprisingly rich and complicated from a constructive viewpoint. Unpublished work [8] shows that it cannot be proven working within Bishop-style constructive mathematics, but that under the assumption of Ishihara's principle **BD-N** it can [5]. We can sketch a simple proof[3] of this latter fact, in which the complete enough extension of Ishihara's tricks plays a central role.

As we have seen in Example 2.2 the space of all permutations $\mathbf{P}$ is a complete enough metric space. Assume we are given a sequence of reals $(a_n)_{n\geqslant 1}$ such that for every permutation $\sigma \in \mathbf{P}$ the series

$$\sum_{n=1}^{\infty} a_{\sigma(n)}$$

converges. It is straightforward to show that the function $F : \mathbf{P} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$F(\sigma) = \left(\sup_{j\geqslant k} \left| \sum_{n=k}^{j} a_{\sigma(n)} \right| \right)_{k\geqslant 1}$$

is well defined, strongly extensional and that it is sequentially continuous under the assumption of **LPO**. Therefore, by Corollary 3.4 it is sequentially continuous. At this point **BD-N** comes into play. We are not going to repeat its definition, but remind the reader that under the assumption of **BD-N** every sequentially continuous map on a countable space is point-wise continuous [7]. Since we have shown that F is sequentially continuous, we can, with **BD-N** assume that it is point-wise continuous.

So for an arbitrary $\varepsilon > 0$ there exists N such that if $\overline{\sigma}N = \overline{\mathrm{id}}N$, i.e. if a permutation σ agrees with the identity on the first N places (i.e. close in Baire space), then

$$\|F(\sigma) - F(\mathrm{id})\|_{\sup} < \frac{\varepsilon}{4} .$$

Since the series is convergent, we may also assume that N is large enough such that

$$\left| \sum_{n=N}^{j} a_n \right| < \frac{\varepsilon}{4} , \tag{2}$$

for all $j \geqslant N$.

We want to show that $\sum_{i=N}^{j} |a_i| \leqslant \varepsilon$ for all $j \geqslant N$. To see this, assume that there is $j \geqslant N$ such that $\sum_{i=N}^{j} |a_i| > \varepsilon$ then there must be $i_1, \dots, i_m$ such that $a_{i_1}, \dots, a_{i_m}$ and

$$|a_{i_1} + \cdots + a_{i_m}| > \frac{\varepsilon}{2} . \tag{3}$$

[3]Some of the details of this proof, and in particular the definition of the function F, are due to D. S. Bridges

Let τ be any permutation that equals the identity on the first N places, and such that $\tau(N+\ell) = i_\ell$ for $1 \leqslant \ell \leqslant m$. By the choice of N

$$\|F(\tau) - F(\mathrm{id})\|_{\sup} < \frac{\varepsilon}{4} \ .$$

Together with Equation 2 this means that

$$\left| \sum_{n=N}^{N+m} a_{\tau(n)} \right| < \frac{\varepsilon}{2} \ ;$$

a contradiction to Equation 3, and therefore we are done.

4.3 Constructive Reverse Mathematics

A popular exercise in first year mathematics is to prove that there is a bijection between $[0,1)$ and $(0,1)$ and that therefore both sets have the same cardinality. Interestingly enough this is not possible constructively. As we will show the existence of such a function is *equivalent* to **LPO**.

Proposition 4.3. ***LPO*** *is equivalent to the existence of a strongly extensional bijection* $f : [0,1) \to (0,1)$.

Proof. It is easy to see that with the help of **LPO**

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \\ x & \text{else} \end{cases}$$

is a well-defined, strongly extensional bijection from $[0,1) \to (0,1)$. Conversely assume that such a function f exist. Then $0 < f(0) < 1$. Since f is surjective there exist a, b such that $0 < f(a) < f(0) < f(b) < 1$. We will construct sequences $(a_n)_{n \geqslant 1}$ and $(b_n)_{n \geqslant 1}$ such that

- $a_n < a_{n+1} < b_{n+1} < b_n$
- $f(a_n) < f(0) < f(b_n)$
- $|a_n - b_n| < \left(\frac{1}{2}\right)^n$

Both sequences converge to a limit z. Now, by injectivity, $f(z) \neq f(0)$, so either $f(z) > f(0)$ or $f(z) < f(0)$. W.l.o.g. assume the first alternative holds and let $\varepsilon = f(z) - f(0) > 0$. Using Ishihara's tricks either $|f(a_n) - f(z)| < \varepsilon$ infinitely often or **LPO** holds. Now the first alternative is ruled out, since

$$|f(a_n) - f(z)| = f(0) - f(a_n) + \varepsilon > \varepsilon \ ,$$

and hence **LPO** holds. ∎

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