BRAUER GROUPS: TALK 2

JEREMY BOOHER

In this talk we will prove two theorems about the Brauer groups of schemes beyond $\operatorname{Spec} k$. This is only a taste of the many results found in Grothendieck's Groupe de Brauer [2]. The proofs use one of the fundamental techniques in studying the Brauer group: relating it to étale cohomology. These results can be combined to give a circuitous proof of a fundamental result in local class field theory.

Recall that a ring R is Henselian if it is local and Hensel's lemma holds. This is equivalent to any finite ring over R being a product of local rings. Local fields are familiar examples. Our first result is due to Grothendieck.

Theorem 1. *Let R be a Henselian ring with residue field k. Then the natural map $\operatorname{Br}(\operatorname{Spec} R) \to \operatorname{Br}(\operatorname{Spec} k)$ is an isomorphism.*

The second result is due to Auslander and Brumer, and relates the Brauer group of a DVR with the Brauer group of its field of fractions.

Theorem 2 (Auslander-Brumer)**.** *Let R be a discrete valuation ring with quotient field K and residue field k with absolute Galois group G_k. Then*

$$0 \to \operatorname{Br}(\operatorname{Spec} R) \to \operatorname{Br}(\operatorname{Spec} K) \to \operatorname{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \to 0$$

is an exact sequence.

Note that $\operatorname{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$ is the dual of $\operatorname{Gal}(k^{\text{sep}}/k)$. It is often denoted $X(G_k)$. Putting these together lets us calculate the Brauer group of a local field.

Corollary 3. *Let K be a local field, a field complete with respect to a discrete valuation with finite residue field k. Then $H^2(\operatorname{Gal}_K, (K^{\text{sep}})^\times) = \operatorname{Br}(K) = \mathbb{Q}/\mathbb{Z}$.*

Proof. Let R be the ring of integers in K. The Auslander-Brumer theorem applies to R, giving a short exact sequence

$$0 \to \operatorname{Br}(\operatorname{Spec} R) \to \operatorname{Br}(\operatorname{Spec} K) \to \operatorname{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \to 0.$$

We know that $\operatorname{Br}(\operatorname{Spec} k) = 0$ as the residue field is finite: we showed last time that every finite division algebra is a field. Theorem 1 then shows that $\operatorname{Br}(\operatorname{Spec} R) = 0$, so $\operatorname{Br}(\operatorname{Spec} K) = \operatorname{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$. But the absolute Galois group of a finite field is $\hat{\mathbb{Z}}$, so $\operatorname{Br}(\operatorname{Spec} K) = \mathbb{Q}/\mathbb{Z}$. □

1. The Brauer Group and Étale Cohomology

Just as the topological Brauer group is related to $H^2(X, \mathcal{O}_X^\times)$, the Brauer group of a scheme is related to the étale cohomology group $H^2(X_{et}, \mathbb{G}_m)$. However, the relationship is not as tight as in the topological setting. The main result is the following theorem.

Theorem 4. *Let X be a quasi-compact scheme with the property that every finite subset is contained in an open affine set. Then there is a natural injective homomorphism $\operatorname{Br}(X) \hookrightarrow H^2(X_{et}, \mathbb{G}_m)$.*

Quasiprojective schemes over affine schemes have this property. The group $H^2(X_{et}, \mathbb{G}_m)$ is called the cohomological Brauer group, and denoted $\operatorname{Br}'(X)$.

Date: February 04, 2013.

Proof. This is Theorem IV.2.5 in Milne's Étale Cohomology [4]. The proof is very similar in spirit (if not in technical details) to the topological result in the last talk, so it will only be sketched.

The first step is to show that the sequence

$$0 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 0$$

is an exact sequence in the étale topology. This relies on a version of the Noether-Skolem theorem. Then one considers the connecting homomorphism in the long exact sequence of étale cohomology

$$\delta : H^1(X_{et}, \mathrm{PGL}_n) \to H^2(X_{et}, \mathbb{G}_m)$$

As in the topological case, the cohomology of GL_n and PGL_n is non-Abelian étale cohomology, defined using cocycles. The technical conditions on X are to ensure that the derived functor cohomology matches the étale Čech cohomology, so we may describe the map to $H^2(X_{et}, \mathbb{G}_m)$ by a map of cocycles. The hypotheses on X may be relaxed by using a more general theory of non-Abelian cohomology as discussed in Milne.

Next, one identifies rank n^2 Azumaya algebras with $H^1(X_{et}, \mathrm{PGL}_n)$. Since the automorphism sheaf of Mat_n is PGL_n by the generalization of Noether-Skolem and Azumaya algebras are étale locally Mat_n, one can obtain a cocycle from an Azumaya algebra (details are found in the section of twisted forms in III.4). Furthermore, Azumaya algebras which are endomorphism algebras of locally free modules of rank n are the image of the map $H^1(X_{et}, \mathrm{GL}_n) \to H^2(X_{et}, \mathbb{G}_m)$ just as in the topological case.

Therefore on can map Azumaya algebras to $H^2(X_{et}, \mathbb{G}_m)$ by combining these maps for all n. A calculation with cocycles, similar to the one in the topological case, shows that this map turns the tensor product of Azumaya algebras into the product in $H^2(X_{et}, \mathbb{G}_m)$. The trivial Azumaya algebras are the image of the $H^1(X_{et}, \mathrm{GL}_n)$, so this factors to give an injective map from the Brauer group $\mathrm{Br}(X)$ to $H^2(X_{et}, \mathbb{G}_m)$. □

The same argument as in the topological case shows the Brauer group is torsion (at least provided X has finitely many connected components). There exist singular schemes where $H^2(X_{et}, \mathbb{G}_m)$ is not torsion, so the Brauer group is not identified with the cohomological Brauer group in all cases. According to Milne, there are no known examples where $\mathrm{Br}(X)$ is not $\mathrm{Br}'(X)[tors]$. In many familiar cases, they are provably the same, such as for smooth schemes and for local rings of dimension at most one [4, IV.2.15,17]. We will prove the case of Henselian local rings in the next section.

Example 5. We already have one case where we know the cohomological Brauer group equals the Brauer group: that of $\mathrm{Spec}\, k$. By analyzing central simple algebras over a field, we determined that $\mathrm{Br}(k) = H^2(G_k, (k^{\mathrm{sep}})^\times)$. On the other hand, $\mathrm{Br}(\mathrm{Spec}\, k)$ injects into $H^2((\mathrm{Spec}\, k)_{et}, \mathbb{G}_m)$ which equals $H^2(G_k, (k^{\mathrm{sep}})^\times)$ by Grothendieck's Galois theory.

2. The Brauer Group of Henselian Rings

We aim to understand the Brauer group of Henselian local rings. The main result will be following: our proof is adapted from Milne [4, IV.2].

Theorem 6. *Let R be a Henselian ring. Then* $\mathrm{Br}(\mathrm{Spec}\, R) = \mathrm{Br}'(\mathrm{Spec}\, R)$.

The key is understanding when an element lies in the image.

Proposition 7. *Let A be a local ring, $X = \mathrm{Spec}\, A$ and γ an element of the cohomological Brauer group $\mathrm{Br}'(X)$. Then γ lies in the image of $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ if and only if there is a finite étale surjective map $Y \to X$ such that γ maps to 0 in $\mathrm{Br}'(Y)$.*

Proof. We may pass to a further étale extension, so $Y = \mathrm{Spec}\, B$ may be assumed to be a Galois cover of X.[1] Recall the Hochsilde-Serre spectral sequence for this cover says there is a spectral sequence

[1]This means the automorphisms of Y over X act simply transitively on fibers.

with $E_2^{p,q} = H^q(G, H^p(Y_{et}, \mathbb{G}_m))$ that converges to $H^n(X_{et}, \mathbb{G}_m)$. This spectral sequence follows from Grothendieck's composition of functors spectral sequence plus realizing that $H^p(G, \Gamma(Y, I)) = 0$ for any injective sheaf because this can be interpreted as Čech cohomology of I for the cover Y over X. Now as B is a Galois cover of local ring A, it is semilocal and hence we know that $H^1(Y_{et}, \mathbb{G}_m) = 0$ by interpretting it as the Picard group. Therefore $E_2^{1,q} = 0$ for all q, and the E_2 page looks like

$$\begin{array}{ccc} H^2(G, H^0(Y_{et}, \mathbb{G}_m)) & 0 & \cdots \\ \\ H^1(G, H^0(Y_{et}, \mathbb{G}_m)) & 0 & H^1(G, H^2(Y_{et}, \mathbb{G}_m)) \\ \\ H^0(G, H^0(Y_{et}, \mathbb{G}_m)) & 0 & H^0(G, H^2(Y_{et}, \mathbb{G}_m)) \end{array}$$

We have that $E_\infty^{1,1} = 0$, $E_2^{0,2} = E_\infty^{0,2}$ and $E_3^{2,0} = E_2^{2,0}$ because of the vanishing column. But then $E_\infty^{2,0} = \ker(E_2^{2,0} \to E_3^{0,3})$, so $E_\infty^{2,0}$ is a submodule of $E_2^{2,0}$. The convergence of the spectral sequence, plus the vanishing of $E_\infty^{1,1}$, gives the short exact sequence

$$0 \to H^2(G, H^0(Y_{et}, \mathbb{G}_m)) \to H^2(X_{et}, \mathbb{G}_m) \to E_\infty^{2,0} \to 0.$$

The hypothesis that an element γ is in $\ker(H^2(X_{et}, \mathbb{G}_m) \to H^2(Y_{et}, \mathbb{G}_m)$ means that it maps to 0 in $H^0(G, H^2(Y_{et}, \mathbb{G}_m))$, so such elements are automatically coming from an element $\gamma' \in H^2(G, H^0(Y_{et}, \mathbb{G}_m))$. Conversely, every such element maps to 0 in $E_\infty^{2,0} \subset H^0(G, H^2(Y_{et}, \mathbb{G}_m))$. But an element $H^2(G, H^0(Y_{et}, \mathbb{G}_m))$ can be represented by a two cocycle with coefficients in $B^\times$. Just as in the case of central simple algebras, we can use this cocycle to write down the multiplication law on an Azumaya algebra over X that splits over Y, and conversely. □

We also need a lemma about étale covers of Henselian local rings.

Lemma 8. *Let R be a Henselian local ring with residue field k. There is an equivalence between étale covers of R and étale covers of k given by restricting to the special fiber.*

Proof. Let K be the field of fractions of R. Given an étale extension l of k, a finite separable extension, pick a primitive element α. Lift the minimal polynomial of α to a polynomial $f(x) \in R[x]$, and adjoin a root to obtain a finite extension L of K. Let S be the integral closure of R in L. Because R is Henselian and S is finite over R, S is a product of local rings. It is also an integral domain, so S is local with residue field l. Finally S is an étale extension of R because L/K is separable as $f(x)$ has distinct roots in the residue field. □

Corollary 9. *With the same notation, $H^2((\operatorname{Spec} R)_{et}, \mathbb{G}_m) = H^2((\operatorname{Spec} k)_{et}, \mathbb{G}_m)$.*

We can now prove Theorem 6. Let R be a Henselian ring, k its residue field, and K its field of fractions. Let γ be an element of $\operatorname{Br}'(\operatorname{Spec} R)$, $\mathfrak{m}$ the closed point of $\operatorname{Spec} R$, with residue field k. As $H^2((\operatorname{Spec} R)_{et}, \mathbb{G}_m) = H^2((\operatorname{Spec}(k))_{et}, \mathbb{G}_m) = \operatorname{Br}(k)$, there is a corresponding central simple algebra over k. By the theory of central simple algebras, this is split over some finite separable extension l/k. This corresponds to an étale cover $\operatorname{Spec} S \to \operatorname{Spec} R$. Then the commuting diagram

$$\begin{array}{ccc} H^2((\operatorname{Spec} S)_{et}, \mathbb{G}_m) & \xrightarrow{\sim} & \operatorname{Br}(l) \\ \uparrow & & \uparrow \\ H^2((\operatorname{Spec} R)_{et}, \mathbb{G}_m) & \xrightarrow{\sim} & \operatorname{Br}(k) \end{array}$$

shows that γ maps to 0 in $H^2((\operatorname{Spec} S)_{et}, \mathbb{G}_m) = \operatorname{Br}'(\operatorname{Spec} S)$. Then apply the proposition. □

Theorem 1 is immediate: both R and its quotient are Henselian, and

$$\mathrm{Br}'(\operatorname{Spec} R) = H^2((\operatorname{Spec} R)_{et}, \mathbb{G}_m) = H^2((\operatorname{Spec} k)_{et}, \mathbb{G}_m) = \mathrm{Br}'(\operatorname{Spec} k).$$

3. The Auslander-Brumer Theorem

The first proof that the Brauer group of a discrete valuation ring equals the Brauer group of its residues is due to Auslander and Brumer [1]. It is of a commutative algebra flavor, and uses Galois cohomology instead of the Azumaya algebras used by Grothendieck. We first need a compatibility theorem between the Brauer group of a discrete valuation ring and the cohomological definition that Auslander and Brumer study. Throughtout, R is a discrete valuation ring, K is its field of fractions, L is the maximal unramified extension of K, S the integral closure of R in L, and $G = \mathrm{Gal}(L/K)$.

Proposition 10. *With the previous notation,* $\mathrm{Br}(\operatorname{Spec} R) = H^2(G, S^\times)$.

Proof. This uses an explicit understanding of Galois cohomology over a DVR and its relation to étale cohomology. This is presented in detail in Stein's notes on Galois cohomology [5], which follow Mazur's discussion of Galois cohomology of number fields. In particular, one shows that $H^2((\operatorname{Spec} R)_{et}, \mathbb{G}_m) = H^2(G, S^\times)$ by combining Theorem 27.6 with the interpretation of the étale sheaf $\mathbb{G}_m$ as a Galois module over the DVR in example 23.5. Thus $\mathrm{Br}'(\operatorname{Spec} R) = H^2(G, S^\times)$. The cohomological Brauer group equals the Brauer group in this situation: we proved this only for Henselian rings, but it holds more generally for local rings of dimension 1 [4, IV.2.17]. □

Furthermore, we need a finer result on the splitting of central simple algebras.

Proposition 11. *Every central simple algebra over a non-archimedean local field K is split by an unramified extension.*

Proof. This is a relatively standard step in the approach to local class field theory via Brauer groups - additional details are found in Milne [3, IV.4]. The idea is to mimic basic algebraic number theory in the ring of integers of a division algebra. Let D be a division algebra over K. Let $\mathcal{O}_K$ be the ring of integers and k the residue field of K. From the theory of central simple algebras, we know that $[D : K] = n^2$. One checks that there is an extension of the valuation to D, defined by considering the field extension of K generated by an element of D, and then defines the ring of integers $\mathcal{O}_D$ and its maximal ideal $\mathfrak{m}_D$ as usual. Furthermore, we know that $l = \mathcal{O}_D/\mathfrak{m}_D$ is a finite division algebra and hence a field. Picking a primitive element and lifting to $\alpha \in D$, $L = K(\alpha)$ is a subfield of D. Thus $f = [l : k] = [L : K] \leq n$, since the maximal subfield in D is of dimension n. Since $K(\alpha)$ is unramified over K, we just need to show that $K(\alpha)$ splits D. By the theory of central simple algebras, this happens if L is maximal, ie if $f = n$.

We can also look at the ramification of this division algebra: the usual proofs go through and show that $\mathfrak{m}_D^e = \mathfrak{m}_K \mathcal{O}_D$ for some integer e. The ramification degree e satisfies $e \leq n$ by considering how the valuation extends. By filtering $\mathcal{O}_D \supset \mathfrak{m}_D \supset \mathfrak{m}_D^2 \supset \ldots \supset \mathfrak{m}_K \mathcal{O}_D$, we see that each quotient has dimension f over k, while the chain is of length e. Thus $\mathcal{O}_D/\mathfrak{m}_K\mathcal{O}_D$ has dimension ef. Since D is of dimension n^2 over K, we see that $ef = n^2$. Thus $e = f = n$. □

Now we construct a short exact sequence of Galois cohomology.

Proposition 12. *There is a split short exact sequence*

$$0 \to H^2(G, \mathcal{O}_L^\times) \to H^2(G, L^\times) \to H^2(G, \mathbb{Z}) \to 0.$$

Proof. For any finite unramified extension M of K, we have a short exact sequence of $G_M = \mathrm{Gal}(M/K)$ modules

$$0 \to \mathcal{O}_M^\times \to M^\times \to \mathbb{Z} \to 0.$$

It is split by choosing a uniformizer $\pi \in M^\times$. Taking the long exact sequence in group cohomology, we see that

$$0 \to H^2(G_M, \mathcal{O}_M^\times) \to H^2(G_M, M^\times) \to H^2(G_M, \mathbb{Z}) \to 0$$

is exact: the other degrees do not contribute as the short exact sequence was split. Now take the limit over all unramified extensions. □

We can now prove the Auslander-Brumer theorem by identifying the cohomology groups in this sequence. Since $H^2(G, \mathbb{Z}) \simeq H^1(G, \mathbb{Q}/\mathbb{Z})$ (consider the connecting homomorphism in $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$), the rightmost term is naturally $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. Since the extension is the maximal unramified one, G is the Galois group of the residue extension, G_k. Thus $H^2(G, \mathbb{Z}) \simeq \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) = X(\mathbb{G}_k)$. Proposition 10 shows $H^2(G, \mathcal{O}_L^\times) = \mathrm{Br}(R)$. Finally, recall we showed that $H^2(G_K, (K^{\mathrm{sep}})^\times)$ is the Brauer group of a field by taking the limit over finite separable extensions of central simple algebras split by the extension, which were described cohomologically. Since every central simple algebra split over some separable extension, the limit was the Brauer group. Since we know that in this case every central simple algebra splits over an unramified extension, we may use the maximal unramified extension instead and still obtain the Brauer group. This completes the proof. □

References

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