THE CUSPS OF HILBERT MODULAR SURFACES AND CLASS NUMBERS

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Hilbert modular surfaces are a generalization of modular curves, and one of the more concrete examples of Shimura varieties. They are formed by taking the product of two copies of the upper half plane and quotienting by a Hilbert modular group arising from a real quadratic field. This produces a complex surface which is neither smooth nor compact. Hirzebruch showed how to resolve the quotient singularities in 1953. If one tries to compactify by adding in a point at each cusp like one does for a modular curve, the resulting surface is singular. It was not until 1973 that Hirzebruch showed how to resolve the cusp singularities by compactifying in a better way so that the resulting surface is smooth. This is similar to the Bailey-Borel compactification used to construct general Shimura varieties, but is explicit. The construction of the singular surface is discussed in Section 1, and Hirzebruch's resolution of the singularities in Section 2. The key idea is to add more than a single point at infinity. The resolution adds in a ring of projective lines, to be described later, that capture the intuitive notion of ways to approach the cusp.

Once this is done, we are able to see many connections with number theory. Hirzebruch proved many such results [6]: we will focus on one particular aspect of this phenomena in Section 3, the relation between the signature defect and L-functions. This has amusing consequences in terms of class numbers for imaginary quadratic fields. We first illustrate by example.

Example 1. The continued fraction expansion for $\sqrt{11}$, $\sqrt{19}$, and $\sqrt{23}$ are

$$\sqrt{11} = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \dots}} = [3, \overline{3, 6}], \qquad \sqrt{19} = [3, \overline{2, 1, 3, 1, 2, 8}], \qquad \sqrt{23} = [4, \overline{1, 3, 1, 8}]$$

The alternating sums of the numbers arising in the repeating parts of the continued fractions are

$$3 - 6 = -3, \qquad 2 - 1 + 3 - 1 + 2 - 8 = -3, \qquad 1 - 3 + 1 - 8 = -9.$$

The class numbers for the rings of integers of $\mathbb{Q}(\sqrt{-11})$, $\mathbb{Q}(\sqrt{-19})$, and $\mathbb{Q}(\sqrt{-23})$ are

$$h(-11) = 1, \qquad h(-19) = 1, \qquad h(-23) = 3.$$

This is an example of the following, which is a shadow of Theorem 33 which relates the signature defect of the cusp of a Hilbert modular surface to special values of L-functions.

Theorem 2. *Let $p > 3$ be prime with $p \equiv 3 \mod 4$. Suppose that the class number of the ring of integers in $\mathbb{Q}(\sqrt{p})$ is one. Write $\sqrt{p} = [a_0, \overline{a_1, \dots, a_s}]$ with s minimal. Then*

$$h(-p) = \frac{1}{3} \sum_{j=1}^{s} (-1)^j a_j.$$

Section 3 discusses the geometry of each cusp, and expresses the signature in terms of the geometry of the compactification, as determined by continued fractions, and alternately in terms of special values of L-functions. Section 4 gives a conceptual explanation for this connection in terms of generalizations of the Atiyah-Singer Index Theorem and the spectrum of a certain differential operator.

Two general references about Hilbert modular surfaces are [6] and [10]. There is also a Bourbaki summary [5].

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1. Constructing Singular Hilbert Modular Surfaces

1.1. **The Action of** $\mathrm{SL}_2(\mathcal{O}_K)$**.** Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, $\mathcal{O}_K$ its ring of integers, U_K the units of $\mathcal{O}_K$, and $\mathbb{H}$ the upper half plane. Hilbert modular surfaces arise by taking a quotient of $\mathbb{H}^2$ by an action of $\mathrm{SL}_2(\mathcal{O}_K)$, adding the cusps and then carefully resolving the singularities. We treat the first two in this section.

The real quadratic field K has two real embeddings σ_1 and σ_2. For $\alpha \in K$, we will often denote $\sigma_i(\alpha)$ by $\alpha^{(i)}$. Recall that an element of K is called totally positive if it is positive under both embeddings of K into $\mathbb{R}$. Let $\mathrm{GL}_2^+(K)$ denote the subgroup of $\mathrm{GL}_2(K)$ with totally positive determinant. The two embeddings give a natural map

$$\mathrm{GL}_2^+(K) \hookrightarrow \mathrm{GL}_2^+(\mathbb{R}) \times \mathrm{GL}_2^+(\mathbb{R}).$$

Each copy of $\mathrm{GL}_2^+(\mathbb{R})$ naturally acts on $\mathbb{H}$ via fractional linear transformations, so $\mathrm{GL}_2^+(K)$ naturally acts on $\mathbb{H}^2$.

We will mostly be interested the action of the subgroup $\mathrm{SL}_2(\mathcal{O}_K)$ and related groups.

Definition 3. Let $\mathfrak{b}$ denote a fractional ideal in K. We define

$$\mathrm{SL}_2(\mathcal{O}_K \oplus \beta) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) : a, d \in \mathcal{O}_K, b \in \mathfrak{b}^{-1}, c \in \mathfrak{b} \right\}.$$

We would like to take a quotient of $\mathbb{H}^2$ by $\mathrm{SL}_2(\mathcal{O}_K)$ and make it into a nice compact complex analytic surface. This is analogous to constructing the modular curve $X_0(1)$ complex analytically by taking the quotient of the upper half plane by $\mathrm{SL}_2(\mathbb{Z})$. As in that case, the obstacle is that the quotient is missing points, such as the point "at infinity". We fix this by adding them.

The group $\mathrm{GL}_2^+(K)$ also acts on $\mathbb{P}^1(K)$, which embeds in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ via the two embeddings of K. The action is obviously transitive. However, if we restrict to the action of $\mathrm{SL}_2(\mathcal{O}_K)$, there are a finite number of orbits. These are the points we will add to $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathbb{H}^2$, and will be called the cusps for $\mathrm{SL}_2(\mathcal{O}_K)$.

Proposition 4. *The cusps for* $\mathrm{SL}_2(\mathcal{O}_K)$ *are naturally in bijection with the ideal class group of* $\mathcal{O}_K$*.*

Proof. Fractional ideals of K are always of the form (α, β). Multiplying by elements of $K^\times$ produces an ideal in the same ideal class. The fractional ideal (α', β') is the same ideal provided there exists a $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$ with

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \gamma \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Therefore the ideal class group is naturally in bijection with the orbits of $\mathrm{SL}_2(\mathcal{O}_K)$ on $\mathbb{P}^1(K)$. □

Remark 5. The point at infinity corresponds to the point $[1, 0] \in \mathbb{P}^1(K)$, which corresponds to the trivial ideal class.

It will convenient to carry out the construction only for the cusp at infinity. Since every cusp can be send there by an element of $\mathrm{SL}_2(K)$, this is not a serious restriction.

Lemma 6. *Let* $\sigma = [\alpha, \beta]$ *be a cusp and* $\mathfrak{b} = (\alpha, \beta)$ *be a representative for the corresponding ideal class. Then there is a matrix* $A_\sigma \in \mathrm{SL}_2(K)$ *that carries* ∞ *to* σ *and such that*

$$A_\sigma^{-1} \mathrm{SL}_2(\mathcal{O}_K) A_\sigma = \mathrm{SL}_2(\mathcal{O}_K \oplus \mathfrak{b}^2).$$

Proof. As $\mathfrak{b}\mathfrak{b}^{-1} = \mathcal{O}_K$, the element 1 is of the form $\alpha\alpha^* + \beta\beta^*$ for $\alpha^*, \beta^* \in \mathfrak{b}^{-1}$. Take $A_\sigma = \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix}$ and compute. □

1.2. **The Local Ring at Infinity.** We now want to add the cusp at infinity to $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathbb{H}^2$ and make this into a complex analytic space. To specify the complex structure and construct a resolution of the cusp singularity we work locally, so we need only quotient a neighborhood of ∞ by this action. More formally, once one defines a $\mathrm{SL}_2(\mathcal{O}_K)$ invariant notion of distance on $\mathbb{H}^2$, one can break $\mathbb{H}^2$ up into the sets of points which are closest to each cusp. The isotropy group of the cusp at infinity acts on the neighborhood of infinity, while elements outside of this subgroup permute the cusps. Therefore to study the cusp singularity we can simply quotient $\mathbb{H}^2$ by the isotropy subgroup of infinity: the details are spelled out in [10, Section 1.2, 1.3],

A direct calculation shows that the elements of $\mathrm{SL}_2(\mathcal{O}_K \oplus \mathfrak{b}^2)$ that fix ∞ are those of the form $\begin{pmatrix} \epsilon & \mu \\ 0 & \epsilon^{-1} \end{pmatrix}$ for ϵ in the units U_K of $\mathcal{O}_K$ and for $\mu \in \mathfrak{b}^{-2}$. Since the action is unchanged by scaling, it is convenient to represent the isotropy group as

$$\{\begin{pmatrix} \epsilon^2 & \mu \\ 0 & 1 \end{pmatrix} : \epsilon \in U_K, \mu \in \mathfrak{b}^{-2}\}$$

This is an example of the following construction.

Definition 7. A complete module M in K is a finitely generated subgroup of K of full rank. Let U_M^+ denote the group of totally positive elements $\epsilon \in K$ which satisfy $\epsilon M = M$. For a finite index subgroup $V \subset U_M^+$, define

$$G(M,V) = \left\{ \begin{pmatrix} \epsilon & \mu \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2^+(K) : \epsilon \in V, \mu \in M \right\}.$$

Note that fractional ideals are examples of complete modules, and the full rank condition is equivalent to being a lattice in $\mathbb{R}^2$ under the real embeddings. Thus the isotropy group can be expressed as $G(\mathfrak{b}^{-2}, U_K^2)$.

We now work with a complete module $M \subset K$ and finite index $V \subset U_M^+$. Add a point, denoted by ∞, to the topological space $G(M,V)\backslash\mathbb{H}^2$ and denote the union by $\overline{G(M,V)\backslash\mathbb{H}^2}$.

To specify the topology it suffices to give a fundamental system of neighborhoods of ∞. Define a neighborhood of infinity in $\mathbb{H}^2$ by

$$N_\infty(r) = \{(z_1, z_2) \in \mathbb{H}^2 : \mathrm{Im}(z_1)\,\mathrm{Im}(z_2) > r\}.$$

A direct calculation shows that $G(M,V)$ acts on this set. Declare that the sets $(G(M,V)\backslash N_\infty(r)) \cup \infty$ give a fundamental system of neighborhoods of ∞. This defines the notion of continuity at infinity, which in turn allows us to define regular functions.

Definition 8. The local ring at infinity, denoted $\mathcal{O}_{\overline{G(M,V)\backslash\mathbb{H}^2},\infty}$, consists of continuous functions on a neighborhood infinity such that there exists an $r > 0$ so the function is holomorphic when viewed as a function on $N_\infty(r)$.

The local ring at infinity has an explicit description in terms of Fourier series.

Lemma 9. *The local ring at infinity is the ring of Fourier series*

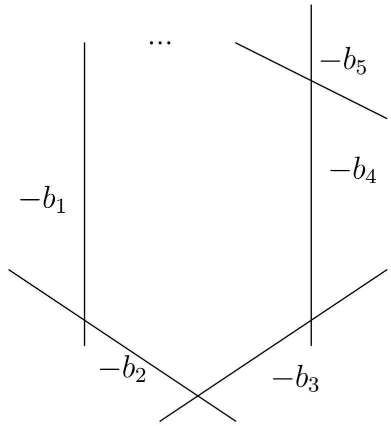
$$a_0 + \sum_{\substack{v \in M^\vee \\ v >> 0}} a_v e^{2\pi i(tr(vz))}$$

that converge on some $N_\infty(r)$ and satisfy $a_v = a_{\epsilon v}$ for all $\epsilon \in V$.

Proof. This is essentially the fact that the transformation $z \to z + \mu$ for $\mu \in M$ is an example of the action by $G(M,V)$, plus a little thought about convergence issues. [10, II.1.1] ☐

The space $\overline{G(M,V)\backslash\mathbb{H}^2}$ will turn out to be a normal complex surface. This is tricky to prove directly: the proof Hirzebruch gives relies on a condition due to Cartan [6, p. 202-203]. It is also possible to deduce this from the structure of the resolution of the singularities [10, II.3.3].

Figure 1. The Structure of a Cusp



Remark 10. There are special points in $\mathbb{H}^2$ where the stabilizer of $\mathrm{SL}_2(\mathcal{O}_K)$ jumps in size which also cause singularities. These elliptic fixed points produce singularities which are much easier to resolve than the cusps: see [10, I.5, II.6]. The resolution is also due to Hirzebruch, but happened years before he resolved the cusps. They too are connected to class numbers, as discovered by Prestel. For example, in the case $K = \mathbb{Q}(\sqrt{D})$ with $D \equiv 1 \mod 4$, the number of elliptic points with a certain type of local behavior is $h(-4D)$ [10, I.5].

Remark 11. As the name suggests, Hilbert modular forms are connected to Hilbert modular surfaces. Analogously to the case of modular curves, Hilbert modular forms are related to differential forms on the Hilbert modular surface. The only complications arise from being careful about the behavior at the singularities [10, I.6, III.3]. This has applications to embedding the surfaces in projective space.

2. Resolving the Cusps

As discussed in the previous section, we need only resolve the singularity at ∞ in $\overline{G(M,V)\backslash\mathbb{H}^2}$. We will construct a complex manifold space $Y(M,V)$ with a holomorphic map $\pi : Y(M,V) \to \overline{G(M,V)\backslash\mathbb{H}^2}$ with the following properties:

1. The map π gives an isomorphism between $Y(M,V) - \pi^{-1}(\infty)$ and $G(M,V)\backslash\mathbb{H}^2$.
2. The exceptional fiber $\pi^{-1}(\infty)$ is a union of copies of $\mathbb{P}^1_{\mathbb{C}}$, labeled $S_1, S_2, \dots, S_r$.
3. The S_i are arranged in a circular configuration as indicated in Figure 1. More precisely, $S_i \cdot S_{i+1} = 1$ and $S_i \cdot S_j = 0$ for $i \neq j \pm 1 \mod r$.
4. The self intersection numbers $S_i \cdot S_i = -b_i$ are related to the pair (M,V) in a way that will be made precise in section 2.3.

The resolution $Y(M,V)$ will be constructed by gluing together copies of $\mathbb{C}^2$ and then quotienting out by M and subsequently by V. This construction is motivated by the theory of toric varieties: the essential idea is that points on the axes in each of the $\mathbb{C}^2$ will correspond to different ways to approach ∞ in $\overline{G(M,V)\backslash\mathbb{H}^2}$.[1] Each choice of basis sees some of the possible ways to approach the cusp, so using multiple copies of $\mathbb{C}^2$ to corresponding to different choices of basis and gluing will produce a smooth compactification.

2.1. **Quotienting by M.**

The complete module $M \subset K$ acts on $\mathbb{H}^2$ via translation: $a \in M$ sends $(z_1, z_2) \mapsto (z_1 + a^{(1)}, z_2 + a^{(2)})$. Given a basis α_1, α_2 for M, we can identify $M\backslash\mathbb{C}^2$ with $\mathbb{C}^\times \times \mathbb{C}^\times$.

[1] This is reminiscent of blowing up, the general method of resolving singularities.

The identification is given by

$$\psi_{\alpha_1,\alpha_2} : (z_1, z_2) \in M\backslash\mathbb{C}^2 \mapsto (u, v) \in \mathbb{C}^\times \times \mathbb{C}^\times$$

where u and v are determined by the condition that for $j = 1, 2$

$$2\pi i z_j \equiv \alpha_1^{(j)} \log(u) + \alpha_2^{(j)} \log(v) \mod 2\pi i M$$

As α_1 and α_2 are a basis, the matrix $\begin{pmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_1^{(2)} & \alpha_2^{(2)} \end{pmatrix}$ is invertible. Taking a logarithm in only defined up to adding a multiple of $2\pi i$, but $2\pi i(z_1, z_2)$ is only defined up to a multiple of $2\pi i M$. Therefore this gives a biholomorphic identification.

The idea behind this identification is to to move the cusp, which is off at infinity in the $\mathbb{C}^2$ picture, into points in the complement of $\mathbb{C}^\times \times \mathbb{C}^\times$ inside $\mathbb{C} \times \mathbb{C}$. The most obvious way for a sequence of points in $\mathbb{H}^2$ to approach the cusp at infinity is for $\mathrm{Im}(z_1)$ or $\mathrm{Im}(z_2)$ to approach infinity, so the real part of $\alpha_1^{(j)} \log(u) + \alpha_2^{(j)} \log(v)$ approaches $-\infty$. This forces $|u|$ or $|v|$ to go to 0, so the sequence is approaching $u = 0$ or $v = 0$ in $\mathbb{C} \times \mathbb{C}$.

The above identification depended on a choice of basis for M. What happens if it is changed? It turn out one natural way to pick a new basis is to let $\beta_1 = \alpha_2$ and $\beta_2 = b\alpha_2 - \alpha_1$ where b is a certain positive integer. Define a map ψ_{β_1,β_2} as above, with coordinates on the image (u', v'), and set $\phi(u, v) = (u^b v, u^{-1})$. A direct calculation shows the following diagram commutes

$$\begin{tikzcd} & \mathbb{C}^\times \times \mathbb{C}^\times \arrow[dd, "\phi"] \\ M\backslash\mathbb{C}^2 \arrow[ur, "\psi_{\alpha_1,\alpha_2}"] \arrow[dr, "\psi_{\beta_1,\beta_2}"'] & \\ & \mathbb{C}^\times \times \mathbb{C}^\times \end{tikzcd} \tag{1}$$

The map ϕ can be extended to a map from $\mathbb{C}^\times \times \mathbb{C}$ to $\mathbb{C} \times \mathbb{C}^\times$ since $b > 0$, but cannot be extended further. What happens to the lines $u = 0$ and $v = 0$ which were related to the cusp? The line $v = 0$ is sent to $u' = 0$, but the lines $u = 0$ and $v' = 0$ are not identified. Changing basis has shown a different aspect of the behavior at the cusp. The following construction will allow us to consider many bases at the same time.

2.2. **Construction.** Let $b_1, b_2, \ldots$ be a sequence of integers such that $b_i \geq 2$ for all i. Let U_i be a copy of $\mathbb{C}^2$ with coordinates u_i and v_i. Let U_i' and U_i'' be the subsets $\{u_i \neq 0\}$ and $\{v_i \neq 0\}$ of U_i. For each k, glue U_k' to U_{k+1}'' via the (holomorphic) map $\phi_k : U_k' \to U_{k+1}''$ defined by

$$\phi_k(u_k, v_k) = \left(u_k^{b_k} v_k, u_k^{-1}\right).$$

An inverse map from $U_{k+1}'' \to U_k'$ is given by sending (u_{k+1}, v_{k+1}) to $(v_{k+1}^{-1}, v_{k+1}^{-b_k} u_{k+1})$.

Definition 12. The topological space $X(b_1, b_2, \ldots)$ is the space obtained by gluing the U_i to U_{i+1} via the map ϕ_i for all i. It will be denoted X when the integers $\{b_k\}$ are clear from context.

Lemma 13. *The topological space X is Hausdorff, and in fact a complex manifold.*

Proof. The key is to directly check that the diagonal is closed [6, p. 204]. Second countability is clear. The U_i give charts with the ϕ_i giving biholomorphic transition functions. □

Denote the curve $v_k = 0$ in U_k by S_k. We will also use S_k to denote the corresponding curve in X. It is also given by the condition $u_{k+1} = 0$ in U_{k+1}. The arrangement of these curves is crucial to the resolution of the cusp singularities.

Proposition 14. *We have $S_k \cdot S_{k+1} = 1$, $S_k \cdot S_j = 0$ if $|k-j| > 1$, and $S_k \cdot S_k = -b_k$.*

Proof. The curve S_k intersects U_k, and U_{k+1} but no other patches because the condition $v_k = 0$ in U_k becomes the condition $u_{k+1} = 0$ in U_{k+1}, which is the complement of U'_{k+1}, while U'_{k-1} is identified with the complement of S_k in U_k. Therefore $S_k \cdot S_{k+1} = 1$ since in the patch U_{k+1} the intersection is simply the (transverse) intersection of $u_{k+1} = 0$ and $v_{k+1} = 0$. Likewise, it is clear that $S_k \cdot S_j = 0$ for $|k-j| > 1$ because the curves do not lie in the same patches.

Finally we turn to the self intersection number. The coordinate function u_{k+1} on U_{k+1} extends to a meromorphic function on X. Since S_k is the locus where $u_{k+1} = 0$, $\operatorname{div}(u_{k+1})|_{U_{k+1}} = S_k$. On U_k, the meromorphic function can be given by $u_k^{b_k} v_k$, which means

$$\operatorname{div}(u_{k+1})|_{U_k} = b_k S_{k-1} + S_k.$$

Therefore $\operatorname{div}(u_{k+1}) = b_k S_{k-1} + S_k + \dots$ where the omitted terms do not intersect U_k. Therefore the intersection product is

$$S_k \cdot \operatorname{div}(u_{k+1}) = b_k S_k \cdot S_{k-1} + S_k \cdot S_k = b_k + S_k \cdot S_k.$$

On the other hand as $\operatorname{div}(u_{k+1})$ is principal the intersection number is 0. Therefore $S_k \cdot S_k = -b_k$. $\square$

Remark 15. It is also possible to give explicit formulas for the transition function from the (u_k, v_k) coordinates on $U_k \cap U_{k+j}$ to the (u_{k+j}, v_{k+j}) coordinates on $U_k \cap U_{k+j}$. Induction shows that

$$u_{k+j} = u_k^{p_j} v_k^{q_k} \quad \text{and} \quad v_{k+j} = u_k^{-p_{j-1}} v_k^{-q_{j-1}}$$

where

$$\begin{pmatrix} p_j & q_j \\ -p_{j-1} & -q_{j-1} \end{pmatrix} = \begin{pmatrix} b_{k+j-1} & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} b_{k+j-2} & 1 \\ -1 & 0 \end{pmatrix} \cdot \dots \cdot \begin{pmatrix} b_k & 1 \\ -1 & 0 \end{pmatrix}.$$

These transition functions make is clear that $U'_1 \cap U''_1 = \{(u_1, v_1) : u_1 v_1 \neq 0\}$ maps bijectively onto $U'_k \cap U''_k$. Therefore X is the union of $U'_1 \cap U''_1$ with the curves $S_1, S_2, \dots$

2.3. **Continued Fractions and Bases for M.** The next step is to describe natural bases for M using continued fractions. As is customary, a continued fraction is denoted by

$$[a_0, a_1, \dots] = a_0 + \frac{1}{a_1 + \dots}.$$

We will also be interested in an alternate form of continued fractions which use subtraction instead of addition.

Definition 16. Given integers $b_1, b_2, \dots,$, define

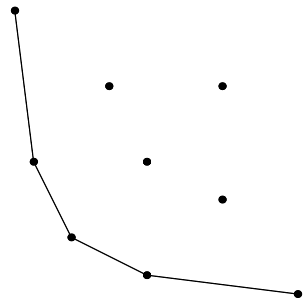
$$[[b_1, b_2, \dots, b_j]] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 \dots \cfrac{1}{b_j}}} \quad \text{and} \quad [[b_1, b_2, \dots]] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \dots}}$$

Just as for ordinary continued fractions, every real number has a expansion of this form, and only rational numbers admit finite ones.

M embeds in $\mathbb{R}^2$ via the two real embeddings of K as a lattice. Denote the set of totally positive elements of M by M_+. Consider the convex hull generated by M_+ in $\mathbb{R}^2_+$: let $\dots, A_{-1}, A_0, A_1, A_2, \dots$ denote the points of M_+ lying on the boundary as in Figure 2. Choose the ordering so $A_{k+1}^{(1)} < A_k^{(1)}$.

Proposition 17. *Two consecutive boundary points A_k and A_{k+1} form a basis for M. There is an integer $b_{k+1} \geq 2$ so that $A_k + A_{k+2} = b_{k+1} A_{k+1}$.*

FIGURE 2. The Convex Hull M_+



Proof. A_k and A_{k+1} certainly form a $\mathbb{Q}$-basis for $M \otimes \mathbb{Q}$. Then any $m \in M_+$ is of the form $cA_k + c'A_{k+1}$ with $c, c' \in \mathbb{Q}$, and by negating m or adding an integral combination of A_k and A_{k+1} we may assume $0 \leq c, c' < 1$ and $c + c' \leq 1$. If $(c, c') \neq (0, 0)$, then m lies outside the convex hull of Min $\mathbb{R}^2_+$, a contradiction.

Now $A_k + A_{k+2}$ is some point in the convex hull, and A_{k+1} lies between them on the boundary. Let b_{k+1} be the maximal integer for which $A_k + A_{k+2} - (b_{k+1} - 1)A_{k+1}$ is totally positive. Then it is a point in the convex hull as well. It must be A_{k+1}, or convexity is violated. □

Furthermore, note that multiplying by a totally positive unit in $\mathcal{O}_K$ that preserves M will preserves the convex hull, and simply scales the two coordinates. It sends boundary points to boundary points, so there is an integer r such that $\epsilon A_k = A_{k+r}$ and hence $b_i = b_{i+r}$ for all i. Therefore the sequence of $\{b_i\}$ is periodic, so it is no loss to truncate and simply consider $b_1, b_2, \ldots$.

The question of how to find the integers b_i and hence the numbers A_i when M is a fractional ideal has been studied classically.

Proposition 18. *If M is generated by 1 and $w_0 \in K$ and w_0 has continued fraction expansion $w_0 = [[b_1, b_2, \ldots]]$, the boundary points A_i are determined by the conditions that $A_1, A_1 = w_0$, and*

$$A_{k+2} = b_{k+1}A_{k+1} - A_k.$$

Proof. This is proven in [10, Section 5]. □

This should not be a big surprise, as the description of the transition function in Remark 15 is closely related to continued fractions: taking $k = 1$ for simplicity

$$\frac{p_j}{q_j} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 \cdots \cfrac{1}{b_j}}}.$$

2.4. **Finishing the Construction.** Now that we have a good choice of bases for M, use the integers b_k to construct $X(b_1, \ldots, b_k, \ldots)$. Recall it is the union of $\{(u_1, v_1) : u_1, v_1 \neq 0\}$ with the curves $S_1, S_2, \ldots$. Map $M \backslash \mathbb{C}^2$ to $\{(u_1, v_1) : u_1, v_1 \neq 0\}$ using ψ_{A_1,A_2}. Because of the commutative diagram (1), this is compatible with mapping $M \backslash \mathbb{C}^2$ to $U'_k \cap U''_k$ using $\psi_{A_k,A_{k+1}}$. Thus there is a biholomorphic map Φ from $M \backslash \mathbb{C}^2$ to $X(b_1, \ldots, b_k) - \cup_j S_j$.

Definition 19. Define $X^+ = X^+(b_1, b_2, \ldots)$ to be $\Phi(M \backslash \mathbb{H}^2) \cup \bigcup_j S_j$.

The final step is to quotient X^+ by V. As the group $G(M, V)$ sits in a split short exact sequence

$$1 \to M \to G(M, V) \to V \to 1$$

we expect this to be a good compactification of $G(M, V) \backslash \mathbb{H}^2$.

Recall that V was a finite index subgroup of U_M^+, a free Abelian group of rank 1, so V is generated by a totally positive unit ϵ with $\epsilon M = M$. By the previous observation that multiplying by ϵ preserves the convex hull of M_+, we conclude that the b_i are periodic with period r. In what follows, assume that at least one b_j is 3, and that $r \geq 3$. These additional assumptions are not completely necessary, but avoid having to consider several exceptional cases which are discussed in [6, 2.2-2.4].

Now V acts on $M\backslash\mathbb{H}^2$ via the usual action, and $\epsilon \in V$ acts on X via sending $(u_k, v_k) \in U_k$ to $(u_k, v_k) \in U_{k+r}$. Because the b_k are periodic with period r, this respects the gluing of the U_k. Furthermore, a direct calculation shows that Φ is equivariant with respect to these actions, so in particular V acts on X^+ as well. Finally, $\epsilon S_j = S_{j+r}$.

Lemma 20. *The action of V on X^+ is free and properly discontinuous.*

Proof. The action on the complement of the S_j is certainly free and properly discontinuous. This also holds for $\cup_j S_j$, which requires a bit of work with the construction [6, Lemma p. 209]. ☐

Because the action is free and properly discontinuous, we can now take the quotient.

Definition 21. Define $Y(M, V)$ to be the quotient of X^+ by V.

Theorem 22. *The space $Y(M, V)$ is a complex manifold with a map $\pi : Y(M, V) \to \overline{G(M, V)\backslash\mathbb{H}^2}$ with the following properties:*

1. *The map π gives an isomorphism between $Y(M, V) - \pi^{-1}(\infty)$ and $G(M, V)\backslash\mathbb{H}^2$.*
2. *The exceptional fiber $\pi^{-1}(\infty)$ is a union of copies of $\mathbb{P}^1_{\mathbb{C}}$, $S_1, S_2, \ldots, S_r$.*
3. *The S_i satisfy $S_i \cdot S_{i+1} = 1$ and $S_i \cdot S_j = 0$ for $i \neq j \pm 1 \mod r$.*
4. *The self intersection numbers $S_i \cdot S_i$ are $-b_k$ where the b_k were the integers used to construct X^+.*

Proof. First, observe the statements about the intersection numbers are obvious because $r \geq 3$ and hence there is no identification of adjacent S_i. (For completeness, Hirzebruch does deals with the case $r = 1, 2$ [6, p. 211].)

The map π is the quotient map coming from the equivariant map Φ^{-1} extended to map the S_j to ∞. It is clear that π gives an isomorphism away from ∞, and the exceptional fiber is the union of the S_j. What is not clear is that π is a map of analytic spaces on the exceptional fiber. One possibility is to check this by hand using the description of the local ring at ∞ for $\overline{G(M, V)\backslash\mathbb{H}^2}$. However, as it was quite difficult to establish this was normal, Hirzebruch avoids this approach. Instead, he invokes a general theorem of Grauert that says the S_j can be blown down to give an isolated normal point in some complex space because of their intersection theory. Then general theory implies that $Y(M, V)$ is a minimal resolution and that the blowdown is singular. The blowdown is topologically isomorphic to $G(M, V)\backslash\mathbb{H}^2$ because $Y(M, V) - \cup_j S_j$ is isomorphic to it, so we use this identification to put a complex structure on $overline G(M, V)\backslash\mathbb{H}^2$ indirectly. Hirzebruch checks the details [6, p. 213]. ☐

Remark 23. Hirzebruch's argument also shows that $Y(M, V)$ is the minimal resolution.

3. Local Invariants of the Cusp and L-functions

The Hirzebruch signature theorem relates the signature of compact $4k$-dimensional manifolds (the signature of the intersection pairing on cohomology) to an expression involving Pontryagin classes. This form of the theorem does not hold for manifolds with boundary, such as $Y(M, V)$. Instead, we will define a signature defect associated to the cusp, which measures how the actual signature deviates from the expected value. These invariants will be related to special values of L-functions. Hirzebruch proved a special case of this connection by comparing an evaluation of certain L-functions with calculations of the signature defect. This led to the investigation and proof of a more general connection which will be discussed in Section 4.

3.1. **Signature Defects.** Let $(X, \partial X)$ be a four dimensional oriented (smooth) manifold with boundary. The intersection pairing gives a symmetric bilinear form on $H^2(X, \partial X, \mathbb{R})$. After diagonalizing this (over $\mathbb{R}$ say), the signature is the number of positive diagonal entries minus the number of negative entries. It will denoted by $\operatorname{sign}(X)$.

Theorem 24 (Hirzebruch Signature Theorem, special case)**.** *Let X be a compact four dimensional smooth manifold without boundary. The signature of X is given by the formula*

$$\operatorname{sign}(X) = \frac{1}{3} p_1(X) \in H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$$

where $p_1(X)$ is the Pontryagin class of tangent bundle of X, and the identification of $H^4(X, \mathbb{Z})$ with $\mathbb{Z}$ is given by Poincaré duality via cupping with the fundamental class $[X]$.

Remark 25. If X is a complex manifold of dimension two, so the tangent bundle is a complex vector bundle, the Pontryagin class of the real tangent bundle can be expressed in terms of the Chern classes of the complex tangent bundle. Because $p_1(X) = -c_2((T_X)_{\mathbb{C}})$ and the complexified tangent bundle splits as $(T_X)_{\mathbb{C}} = T_X \oplus \overline{T_X}$, the properties of Chern classes show that

$$\operatorname{sign}(X) = \frac{1}{3} p_1(X) = \frac{1}{3}\left(c_1(X)^2 - 2c_2(X)\right). \tag{2}$$

We are interested in how this formula fails for the non-compact $Y(M, V)$ considered in the previous section. It will be technically easier to work with a manifold with boundary, so we will cut off the open set $Y(M, V)$ along $\operatorname{Im}(z_1)\operatorname{Im}(Z_2) = r$. Therefore we work with the quotient of

$$\overline{N}_\infty(r) = \{(z_1, z_2) \in \mathbb{H}^2 : \operatorname{Im}(z_1)\operatorname{Im}(z_2) \geq r\}$$

by $G(M, V)$. Note that the boundary N, the set of points with $\operatorname{Im}(z_1)\operatorname{Im}(z_2) = r$, is a principal homogenous space for $G(M, V)$. Now form the quotient of $\overline{N}_\infty(r)$ by $G(M, V)$, adding in the resolution of the cusp in $Y(M, V)$ we constructed in the previous section. We obtain a smooth complex surface $\tilde{Y}(M, V)$ with boundary N.

Lemma 26. *The tangent bundle of $\tilde{Y}(M, V)$ is a pull back of a complex vector bundle E on the quotient $\tilde{Y}(M, V)/N$.*

Proof. See for example the proof of the theorem [6, p. 222] or the discussion [1, p. 137]. The idea is to use the standard hyperbolic metric on $\mathbb{H}^2$ to trivialize the tangent bundle on $\tilde{Y}(M, V)$ that are invariant under $G(M, V)$ on the boundary. The same can be done for the normal bundle. □

The bundle E will allow us to talk about Chern classes despite the manifold having a boundary.

Definition 27. Define $c_i(\tilde{Y}(M, V))$ to be $c_i(E) \in H^{2i}(\tilde{Y}(M, V)/N, \mathbb{Z}) = H^{2i}(\tilde{Y}(M, V), N, \mathbb{Z})$.

Note that despite being defined in terms of E, we can do many calculations involving these classes using the tangent bundle of $\tilde{Y}(M, V)$. For example, to compute the Euler characteristic we can look at the index of a vector field which points in the outward normal direction along the boundary: this computes the Euler characteristic but also descends to a section of E and computes a Chern class. However, if we try to use these classes to compute the signature of the intersection pairing on middle cohomology, the Hirzebruch signature theorem does not hold in general.

Definition 28. The signature defect $\delta(M, V)$ is defined to be

$$\delta(M, V) = \frac{1}{3}\Big(c_1(\tilde{Y}(M, V))^2 - 2c_2(\tilde{Y}(M, V))\Big) - \operatorname{sign}(\tilde{Y}(M, V), N).$$

Again we identity $H^4(\tilde{Y}(M, V), N, \mathbb{Z})$ with $\mathbb{Z}$ using Poincaré duality.

Remark 29. This is a special instance of a more general phenomena. Given a global statement for closed manifolds like the Hirzebruch signature theorem, one can define a defect for nice boundary manifolds N as follows. Pick a manifold X with boundary N, and compute the discrepancy in the global statement for the manifold X. This tends to be independent of the choice of X, since any two choices can be glued together along N after reversing the orientation on one to obtain a closed manifold with defect 0. We have simply picked a particular X.

The amazing fact is that the signature defect depends on the geometry of the cusp at infinity.

Theorem 30. *With the integers $b_1, \dots, b_r$ defines as before for the pair (M, V),*

$$\delta(M, V) = -\frac{1}{3}(b_1 + b_2 + \ldots + b_r) + r.$$

We first compute the signature of $\tilde{Y}(M, V)$, then the Chern classes.

Lemma 31. *The signature of $(\tilde{Y}(M, V), N)$ is $-r$.*

Proof. Looking at the construction of $\tilde{Y}(M, V)$, we see that $S_1 \cup S_2 \cup \ldots \cup S_r$ is a deformation retract of $\tilde{Y}(M, V)$. We wish to compute the signature using Poincaré duality to convert the relative cohomology to homology. This is a union of r copies of $\mathbb{P}^1_{\mathbb{C}}$ arranged cyclically which has second Betti number r. We will show that $S_1, \dots, S_r$, viewed as homology classes, are generators. Theorem 22 implies the intersection matrix for $S_1, \dots, S_r$ by is

$$\begin{pmatrix} -b_1 & 1 & 0 & \ldots & 0 & 1 \\ 1 & -b_2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -b_3 & 1 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & \ldots & 0 & 1 & -b_r \end{pmatrix}$$

This is easily checked to be negative definite, so the classes S_i generate the cohomology group and hence the signature of the intersection pairing is $-r$. □

Lemma 32. *The Chern classes are $c_1(\tilde{Y}(M, V)) = S_1 + S_2 + \ldots + S_r \in H^2(\tilde{Y}(M, V), N, \mathbb{Z})$ and $c_2(\tilde{Y}(M, V)) = r$.*

Proof. The second Chern class is the top one, so it corresponds to the Euler class of the bundle E. But we know is the same as the Euler characteristic of the tangent bundle of $\tilde{Y}(M, V)$, and likewise is the Euler characteristic of $\tilde{Y}(M, V)$. As this retracts onto $S_1 \cup S_2 \cup \ldots \cup S_r$, we just need to compute this one. Using that compactly supported Euler characteristic is additive over disjoint unions, it follows that the Euler characteristic is $2r - r = r$.

Now the first Chern class corresponds to an element $z \in H_2(\tilde{Y}(M, V), \mathbb{Z})$. We will denote the classes in $H_2(\tilde{Y}(M, V), \mathbb{Z})$ generated by the curves S_j by the same symbol. Now (one version of) the adjunction formula[2] states that for any smooth compact curve C on a complex surface X,

$$\chi(C) = c_1(X) \cdot C - C \cdot C$$

Applying this to the curve S_j, we see that

$$z \cdot S_j - S_j \cdot S_j = 2$$

since the Euler characteristic of $S_j \simeq \mathbb{P}^1_{\mathbb{C}}$ is 2. Since the intersection pairing is given by an invertible matrix, these r conditions uniquely determine z. But they are satisfied by taking $z = S_1 + S_2 + \ldots + S_r$ using the information in Theorem 22. □

[2]A discussion of this version is in [6, 0.6].

Finally using the lemmas and Theorem 22, we calculate

$$\delta(M,V) = \frac{1}{3}\left((S_1+S_2+\ldots+S_r)^2 - 2r\right) - (-r) = \frac{1}{3}\left(b_1+b_2+\ldots+b_r+2r-2r\right)+r$$
$$= -\frac{1}{3}(b_1+b_2+\ldots+b_r)+r$$

This completes the proof of the theorem. □

3.2. **Shimizu L-functions.** Let M be a complete module in K, and V a finite index subgroup of U_M^+, the group of totally positive units which satisfy $\epsilon M = M$. Shimizu defined and studied the L-functions

$$L(M,V,s) = \sum_{\mu \in (M-\{0\})/V} \frac{\operatorname{sign} N(\mu)}{|\,\mathrm{N}(\mu)|^s}.$$

Information can be found in [8], but all we will need is presented and mostly proven in [10, III.2]. This L-function can be expressed as a contour integral of the derivative of a non-holomorphic Eisenstein series $E(z,s)$ multiplied by some Γ-factors, which gives it a meromorphic continuation to $\mathbb{C}$. The functional equation for $E(z,s)$ gives a functional equation for $L(M,V,S)$, related to the L-function for the dual module $M^\vee$. The values of these L-functions at $s=1$ had been studied before Hirzebruch. Meyer had calculated them using the Kronecker limit formula, and expressed the answer in terms of the transformation law for the Dedekind η function. Siegel gives an exposition of this result in English [9, Equation 120-122]. This L-value is connected to the numbers b_i associated to the pair (M,V) in the previous section. This reformulation is spelled out by Zagier [11]: the final evaluation is that

$$L(M,V,1) = \frac{\pi^2}{3\sqrt{\Delta_M}} \sum_{i=1}^{r} (b_i - 3) \tag{3}$$

where Δ_M is the volume of the lattice obtained by embedding M in $\mathbb{R}^2$. In light of Theorem 30, we conclude

Theorem 33. *With the notation as above,* $L(M,V,1) = -\frac{\pi^2}{\sqrt{\Delta_M}}\delta(M,V)$.

This is a surprising connection between L-functions and the geometry of the cusp, but this proof is unsatisfying because the two quantities are not directly related. Inspired by this fact, it was generalized and proven in a more conceptual way by Atiyah, Donnelly and Singer using a method based on the Atiyah-Singer index theorem. We outline the connection in Section 4.

Remark 34. These sorts of formulas also led Zagier and Shintani to develop methods of computing the values of L-functions for totally real fields in terms of a generalization of the convex hull of M_+. For an example and references, see [10, III.2.3].

3.3. **An Application to Class Numbers.** Let $p \equiv 3 \mod 4$ be a prime, not equal to 3, and suppose the class number of $\mathbb{Q}(\sqrt{p})$ is 1. Then the Hilbert modular surface has only one cusp, with isotropy group $G(\mathcal{O}_K, U_K^+)$ where $\mathcal{O}_K = \mathbb{Z}[\sqrt{p}]$ is the ring of integers and U_K^+ are the totally positive units which preserve $\mathcal{O}_K$. We will now derive Theorem 2 about the class number of $\mathbb{Q}(\sqrt{-p})$ using Theorem 33. We rephrase the expression on the right using the theory of continued fractions, and evaluate the L function on the left using the analytic class number formula.

We wish to rewrite the expression $-\frac{1}{3}(b_1+b_2+b_3+\ldots+b_r)+r$ appearing in Theorem 30. Recall that $\sqrt{p} = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \ldots}}$.

Lemma 35. *Let* $\sqrt{p} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} = [a_0, \overline{a_1, a_2, \dots, a_s}]$. *Then*

$$(b_1 + b_2 + b_3 + \ldots + b_r) - 3r = \sum_{j=1}^{s} (-1)^j a_j.$$

Proof. This is elementary: a direct calculation shows that

$$[a_0, a_1, z] = [[a_0 + 1, \underbrace{2, 2, \ldots, 2}_{a_1 - 1}, z + 1]]$$

Therefore the alternate continued fraction expansion for $\sqrt{p}$ looks like

$$[[a_0 + 1, \overline{\underbrace{2, 2, \ldots, 2}_{a_1 - 1}, a_2 + 2, \underbrace{2, 2, \ldots, 2}_{a_3 - 1}, \ldots, a_s + 2}]].$$

(It is a general fact that the continued fraction for $\sqrt{p}$ always has even period.) Therefore

$$\sum_{i=1}^{r} (b_i - 3) = \sum_{j=1}^{s} (-1)^j (a_j - 1) = \sum_{j=1}^{s} (-1)^j a_j$$

again using s is even. □

Lemma 36. *If* $h(-p)$ *denotes the class number of* $\mathbb{Q}(\sqrt{-p})$,

$$L(\mathcal{O}_K, U_K^+, 1) = \frac{\pi^2}{2\sqrt{p}} h(-p).$$

Proof. First, recall that since $p \equiv 3 \mod 4$ the norm of the fundamental unit of $\mathcal{O}_K$ is 1. Therefore if α and α' generate the same ideal in $\mathcal{O}_K$ $N(\alpha)$ and $N(\alpha')$ have the same sign. Define a character χ on ideals of $\mathcal{O}_K$ by setting $\chi((\alpha)) = \mathrm{sign}(N(\alpha))$. Since the class number is one, all ideal are principal, so this suffices. The ideals of $\mathcal{O}_K$ are parametrized by elements of $\mathcal{O}_K$ up to multiplication by units. Since the totally positive units are of index 2, we have

$$L(\mathcal{O}_K, U_K^+, s) = 2 \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{|N(\mathfrak{a})|^s}.$$

The character χ is an example of a genus character (a character of the narrow class group). It corresponds to the decomposition $4p = (-4)(-p)$ of the discriminant of $\mathcal{O}_K$. This means there is a factorization of L-functions [9, Theorem 4]

$$L(\mathcal{O}_K, U_K^+, s) = 2L(\chi_{-4}, s) L(\chi_{-p}, s)$$

where χ_D is the Dirichlet character naturally associated to the quadratic field $\mathbb{Q}(\sqrt{D})$. The analytic class number formula, for imaginary quadratic fields, states that

$$L(\chi_{-D}, 1) = \frac{2\pi}{w_{-D}\sqrt{D}} h(-D)$$

where w_D is the number of units in $\mathcal{O}_{\mathbb{Q}(\sqrt{-D})}$. In particular, as $\mathbb{Z}[i]$ has class number one

$$L(\chi_{-4}, 1) = \frac{\pi}{4} \quad \text{and} \quad L(\chi_{-p}, 1) = \frac{\pi}{\sqrt{p}} h(-p).$$

Combining these, we see that

$$L(\mathcal{O}_K, U_K^+, 1) = \frac{\pi^2}{2\sqrt{p}} h(-p).$$

□

Combining the two lemmas with Theorem 33 gives Theorem 2:

$$h(-p) = \frac{1}{3}\sum_{j=1}^{s}(-1)^j a_j. \tag{4}$$

Example 37. Let $p = 163$. Then $\mathbb{Q}(\sqrt{163})$ has class number one, and

$$\sqrt{163} = [12, \overline{1, 3, 3, 2, 1, 1, 7, 1, 11, 1, 7, 1, 1, 2, 3, 3, 1, 24}].$$

Furthermore,

$$\frac{1}{3}(1-3+3-2+1-1+7-1+11-1+7-1+1-2+3-3+1-24) = 3.$$

Therefore we verify the famous fact that $\mathbb{Q}(\sqrt{-163})$ has class number one.

Remark 38. A more general version of Lemma 36 can be proven using the same technique if there are multiple cusps. Take the sum of the signature defects for all of the cusps: a little more care shows that after converting into a sum of Shimizu L-functions using Theorem 33, these L-functions piece together to the L-function for the genus character. The proof continues as before.

4. Shimizu L-functions and the Atiyah-Patodi-Singer Index Theorem

This section will give a high level overview of the connection between special values of the Shimizu L-function and the signature defect of the cusp by relating them to a "spectral" invariant η arising in a version of the Atiyah-Singer Index Theorem for manifolds with boundary. The full details are found in [1], which relies heavily on the Atiyah-Patodi-Singer Index Theorem for manifolds with boundary ([2] [3] [4]). Before we discuss this, let us first sketch how the Atiyah-Singer Index Theorem for manifolds without boundary specializes to the Hirzebruch signature theorem.

4.1. **An Application of the Atiyah-Singer Index Theorem.** Information about the Atiyah-Singer Index Theorem and its applications to appears in many sources: a random one which fully explains the connection with the Hirzebruch signature theorem is [7] . Let X be a compact n-manifold (without boundary), and let E and F be smooth vector bundles on X. Let $D : E \to F$ be an elliptic differential operator. The analytic index of D is the index as a Fredholm operators:

$$i_a(D) = \dim\ker D - \dim\ker D^*.$$

The topological index of D is

$$i_t(D) = (-1)^n \varphi^{-1}\,\mathrm{ch}(\sigma(D))\,\mathrm{td}(T^*(X)_{\mathbb{C}}) \cdot [X]$$

where φ is the Thom isomorphism, ch is the Chern character, $\sigma(D)$ is a bundle on TX defined using the symbol of D, and td denotes the Todd class.

Theorem 39 (Atiyah-Singer)**.** *The topological index of D equals the analytic index of D.*

By picking appropriate vector bundles and operator D, we can recover the Hirzebruch signature theorem. We will simply do this for a complex surface, but the same argument works for the Hilbert signature theorem for $4k$-manifolds [7, Theorem V.3.4].

Let X be a complex surface viewed as a four dimensional real manifold. Let $E^\bullet = \Omega^\bullet_X \otimes \mathbb{C}$ and $A^\bullet = C^\infty(X, E^\bullet)$. Denote the exterior derivative and Hodge star by $d : E^k \to E^{k+1}$ and $* : E^k \to E^{4-k}$. Define the codifferential $\delta = *d* : E^{k+1} \to E^k$. Hodge theory says that

$$H^*(X, \mathbb{C}) = \ker(d\delta + \delta d) = \ker(\delta + d)$$

Furthermore, remember that the inner product for $\alpha, \beta \in A^2$ by definition is given by

$$(\alpha, \beta) = \int \alpha \wedge *\beta.$$

Consider the operator $D = \delta + d$ and the involution $\tau : E^k \to E^{4-k}$ with $\tau(\alpha) = (-1)^{\frac{k(k-1)}{2}+1} * \alpha$. Let $E_+^\bullet$ and $E_-^\bullet$ be the $+1$ and -1 eigenspaces for τ. A direct calculation shows τ anti-commutes with D, so D preserves the eigenspaces and so D decomposes

$$D = D^+ + D^- \quad \text{where} \quad D^+ : E_+^\bullet \to E_-^\bullet, D^- : E_-^\bullet \to E_+^\bullet.$$

The two operators are adjoint. Now Hodge theory tells us that

$$\ker D = \left[H_0(X,\mathbb{C}) \oplus H^4(X,\mathbb{C})\right] \oplus \left[H^1(X,\mathbb{C}) \oplus H^3(X,\mathbb{C})\right] \oplus H^2(X,\mathbb{C})$$

The involution τ respects this decomposition. For dimension reasons, the first two terms split into equal dimension eigenspaces, so the analytic index $\dim\ker D^+ - \dim\ker D^-$ can be computed just on $H^2(X,\mathbb{C})$. But if $\alpha \neq 0$ is in the $+1$ eigenspace for $\ker D^+ \cap H^2(X,\mathbb{C})$,

$$\int_X \alpha \wedge \alpha = \int_X \alpha \wedge (*\alpha) = (\alpha,\alpha) > 0.$$

Likewise the inner product is negative definite on $\ker D^- \cap H^2(X,\mathbb{C})$. Therefore the analytic index of D^+ equals the signature of the manifold.

On the other hand, the topological index for D^+ is easy to describe. It is a polynomial in the Pontryagin (equivalently, Chern) classes of X. Unwinding the definition, one sees that $i_t(D^+) = \frac{1}{3}p_1(X) = \frac{1}{3}\left(c_1(X)^2 - 2c_2(X)\right)$. (For a generalization, see [7, Proposition III.6.2].)

The Hirzebruch signature theorem is therefore just a special instance of the Atiyah-Singer Index Theorem.

4.2. **A Signature Theorem for Manifolds with Boundary.** If $(X, \partial X)$ is a manifold with boundary, the Atiyah-Singer index theorem no longer holds. An error term $\eta_A(0)$ comes from the differential operator on the boundary. The connection with number theory arises from the fact that η is a holomorphic function defined as a Dirichlet-like series in terms of the spectrum of the operator A on the boundary. Atiyah, Donnelly, and Singer analyze the particular case when $X = \tilde{Y}(M,V)$ and $A = *d - d*$ on even forms. This is the operator yielding the Hirzebruch signature theorem, up to dualizing via the Hodge star. They show that

$$\operatorname{sign}(X) = \frac{1}{3}p_1(X) - \eta_A(0)$$

where η_A is the Dirichlet-like series defined by

$$\eta_A(s) = \sum_\lambda \operatorname{sign}(\lambda)|\lambda|^{-s}$$

where sum runs over the non-zero spectrum of A acting on the boundary manifold. There are considerable details involved with framing the boundary, establishing a functional equation and analytic continuation for $\eta_A(s)$, and making sure to use the correct variant of Pontryagin (Chern) classes that will be meaningful on a manifold with boundary. The main input to the statement is the techniques used by Atiyah, Patodi, and Singer to prove their general signature theorem (although they require some mild modifications in this particular setting compared to [4]).

The goal is to understand the relationship of η with the quantities considered by Hirzebruch. We use M' to denote the dual fractional ideal with respect to the trace.

Theorem 40. *With the notation above, $\eta_A(0) = L(M', V, s)$.*

Most of the effort in [1] goes into proving this statement. The first step is to realize the boundary as a torus bundle over S^1, and work separately on each fiber using Fourier series. This allows one to decomposes

$$\eta_A(s) = \eta_0(s) + \sum_{\mu \in (M'-0)/V} \eta_\mu(s)$$

which begins to look like a Shimizu L-function. The bulk of the work goes into understanding the behavior of $\eta_\mu(s)$ as $s \to 0$ and obtaining the terms of the Shimizu L-function.

The signature defect Hirzebruch considered is $\eta_A(0)$, which is related to $L(M', V, 0)$ by the generalized signature theorem plus this analysis. Finally, that value is related to $L(M, V, 1)$ using the functional equation.

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