THE WEIL CONJECTURES AND ANALOGUES IN COMPLEX GEOMETRY

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The Weil conjectures are a statement about the zeta function of varieties over finite fields. The desire to prove them motivated the development of étale cohomology, a process begun by Grothendieck and finished by Deligne. The hardest part is an analogue of the Riemann Hypothesis. In section 1 we define the zeta function and state the conjecture. In section 2, we prove the conjecture for elliptic curves following Hasse. Finally in section 3 we explain a proof due to Serre of a complex analogue of the Weil conjectures which is a simplified model of the proof using étale cohomology.

1. Zeta Functions of Varieties and the Weil Conjectures

In number theory, one studies the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

to prove facts about primes. The series converges only for $\operatorname{Re} s > 1$, but there is an extension to a meromorphic function on the entire complex plane. The location of the zeros controls the distribution of primes, and the Riemann hypothesis states that all of the non-trivial zeros have real part $\frac{1}{2}$.

Further work in algebraic number theory led to considering variants for number fields: if K is a number field and $\mathcal{O}_K$ is its ring of integers, the Dedekind zeta function is defined as

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} N(I)^{-s} = \prod_{\mathfrak{p} \subset \mathcal{O}_K \text{prime}} \frac{1}{1-N(\mathfrak{p})^{-s}}.$$

Taking $K = \mathbb{Q}$ recovers the Riemann zeta function. The Euler products are readily reinterpreted in terms of geometry: the prime ideals are the closed points of $\operatorname{Spec} \mathcal{O}_K$ and $N(\mathfrak{p})$ is the size of the residue field. This generalization leads to the notion of an arithmetic zeta function.

Definition 1. Let X be a scheme of finite type over $\mathbb{Z}$. Define

$$\zeta_X(s) = \prod_x \frac{1}{1-N(x)^{-s}}$$

where x runs over all closed points with $N(x)$ the size of the finite residue field.

As before, it is easy to show there is an Euler product obtained by grouping points above different primes:

$$\zeta_X(s) = \prod_p \zeta_{X_p}(s)$$

These local zeta functions are the objects of interest in the Weil conjectures. In $\zeta_{X_p}(s)$, the term $\frac{1}{1-p^{-ns}}$ appears once for each point with residue of size exactly $\mathbb{F}_{p^n}$, so we see

$$\log \zeta_{X_p}(s) = -\sum_{n=1}^{\infty} |E(\mathbb{F}_{p^n})| \log(1-p^{-ns}) = \sum_{n=1}^{\infty} |E(\mathbb{F}_{p^n})| \frac{p^{-ns}}{n}.$$

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The last equality follows by expanding the logarithm in a power series and realizing that the $\mathbb{F}_{p^n}$ points include points with smaller residue fields. We wish to study this quantity: by convention, we take $T = p^{-s}$.

Definition 2. Let V be defined over $\mathbb{F}_q$. Define the (local) zeta function of V over $\mathbb{F}_q$ as

$$Z(V,T) = \exp\left(\sum_{n=1}^{\infty} |V(\mathbb{F}_{q^n})| \frac{T^n}{n}\right).$$

We first calculate the zeta function of $\mathbb{P}^m$ over $\mathbb{F}_q$. By decomposing projective space as $\mathbb{A}^m \cup \mathbb{A}^{m-1} \cup \ldots$, it is clear that $|\mathbb{P}^n(\mathbb{F}_{q^n})| = 1 + q^n + q^{2n} + \ldots + q^{mn}$. Then we calculate

$$\log Z(\mathbb{P}^n, T) = \sum_{n=1}^{\infty}(1 + q^n + q^{2n} + \ldots + q^{mn})\frac{T^n}{n} = \sum_{j=1}^{m}\sum_{n=1}^{\infty}\frac{q^j T}{n} = \sum_{j=1}^{m} -\log(1 - q^j T).$$

Therefore we conclude

$$Z(\mathbb{P}^n, T) = \frac{1}{(1-T)(1-qT)\ldots(1-q^m T)}.$$

Here are several features to note: the zeta function is a rational function, and it is almost symmetric with respect to replacing T by $\frac{1}{q^m T}$:

$$Z(\mathbb{P}^n, \frac{1}{q^m T}) = (-1)^{m+1} q^{\frac{m(m+1)}{2}} T^{m+1} Z(\mathbb{P}^n, T)$$

Taking $T = q^{-s}$, this becomes

$$Z(\mathbb{P}^n, q^{-(m-s)}) = (-1)^{m+1} q^{\frac{m(m+1)}{2} - (m+1)s} Z(\mathbb{P}^n, q^{-s})$$

which is a a functional equation of sorts.

Remark 3. The same sort of argument works for Grassmanians or any variety that can be paved by affines.

There are other classes of varieties where it is possible to calculate the zeta function "by hand." For example, the computation of the case of diagonal hypersurfaces may be found in Chapter 11 and of Ireland and Rosen [1].

In the 1930's, Hasse proved a conjecture due to Artin about the number of points on an elliptic curve over $\mathbb{F}_q$. It is now known as the Hasse bound, and is the key ingredient to describing the zeta function of an elliptic curve defined over $\mathbb{F}_q$. Weil generalized these conclusions to curves of arbitrary genus and formulated the Weil conjectures for higher dimensional varieties.

Theorem 4 (The Weil Conjectures)**.** *Let V be a non-singular projective variety over $\mathbb{F}_q$ of dimension m with Euler characteristic $\chi(V)$. Then*

- *(Rationality) $Z(X,T)$ is a rational function of T. Furthermore,*

$$Z(V,T) = \frac{P_1(T) \cdot P_3(T) \cdot \ldots \cdot P_{2m-1}(T)}{P_0(T) \cdot P_2(T) \cdot \ldots \cdot P_{2m}(T)}$$

  where $P_0(T) = 1 - T$, $P_{2m}(T) = (1 - q^{2m}T)$, and all the polynomials $P_i(T)$ have integer coefficients.
- *(Functional Equation) $Z(V, \frac{1}{q^m T}) = \pm q^{\chi(V)m/2} T^{\chi(V)} Z(V,T)$.*
- *(Riemann Hypothesis) Factor $P_i(T) = (1 - \alpha_1 T)\ldots(1 - \alpha_k T)$. Then $|\alpha_i| = q^{i/2}$.*
- *(Betti numbers) If V is the reduction of a smooth variety over a field which embeds in $\mathbb{C}$, the ith Betti number of the complex points equals the degree of P_i.*

These were known for elliptic curves by the work of Hasse and for all curves by the work of Weil. Dwork proved rationality using methods from p-adic analysis. All but the Riemann hypothesis follow easily from the étale cohomology theory pioneered by Grothendieck. Deligne finally proved the Riemann hypothesis using étale cohomology in 1974.

2. The Weil Conjectures for Elliptic Curves

In this section we review the classic proof, due to Hasse, that elliptic curves satisfy the Weil conjectures. A canonical reference is Chapter V of Silverman [3].

Let E be an elliptic curve defined over $\mathbb{F}_q$, with $q = p^f$, and let l be a prime not equal to p. Recall that an endomorphism ψ of E defines an endomorphism ψ_l of the l-adic Tate module $T_l E = \varprojlim E[l^n]$ and $\deg(\psi) = \det(\psi_l)$ (use the Weil pairing). Recall that there is a Frobenius endomorphism $\phi : E \to E$ defined on points by raising to the qth power. The fixed points of ϕ are exactly $E(\mathbb{F}_q)$. So to understand the point counts in the zeta function, we need to understand ϕ and ϕ_l.

Proposition 5. *The characteristic polynomial of ϕ_l is $T^2 - (1 + q - |E(\mathbb{F}_q)|)T + q$.*

Proof. It suffices to compute the trace and determinant of ϕ_l as an endomorphism of $T_l E \simeq \mathbb{Z}_l^2$. The determinant is the degree of ϕ_l, which is q. An elementary calculation with two by two matrices shows that $tr(\phi_l) = 1 + \det(\phi_l) - \det(1 - \phi_l) = 1 + q - \det(1 - \phi_l)$. The endomorphism $1 - \phi$ is separable (use the invariant differential), so the size of the kernel is equal to its degree. In other words,

$$|E(\mathbb{F}_q)| = \deg(1 - \phi) = \det(1 - \phi_l).$$

Now express the characteristic polynomial in terms of norm and trace. □

Now define $a = 1 + q - |E(\mathbb{F}_q)|$. It is commonly referred to as the trace of Frobenius. This gives a way to count $\mathbb{F}_q$-points using linear algebra. We can extend this to $\mathbb{F}_{q^n}$ points.

Corollary 6. *Let α and β be the complex roots of $T^2 - aT + q$. Then $|E(\mathbb{F}_{q^n})| = q^n + 1 - \alpha^n - \beta^n$.*

Proof. The characteristic polynomial of $1 - \phi_l^n$ is $(T + (\alpha^n - 1))(T + (\beta^n - 1))$. Therefore we see

$$|E(\mathbb{F}_{q^n})| = \deg(1 - \phi^n) = \det(1 - \phi_l^n) = (\alpha^n - 1)(\beta^n - 1) = q^n + 1 - \alpha^n - \beta^n.$$ □

We can now check that the Weil conjectures hold. The zeta function satisfies

$$\begin{aligned} \log Z(E,T) = \sum_{n=1}^{\infty} |E(\mathbb{F}_{q^n})| \frac{T^n}{n} &= \sum_{n=1}^{\infty} (q^n + 1 - \alpha^n - \beta^n) \frac{T^n}{n} \\ &= -\log(1 - qT) - \log(1 - T) + \log(1 - \alpha T) + \log(1 - \beta T). \end{aligned}$$

Therefore the rationality statement holds:

$$Z(E,T) = \frac{(1 - \alpha T)(1 - \beta T)}{(1 - T)(1 - qT)}$$

Since an elliptic curve is a complex torus, the Betti zeroth, first, and second Betti numbers are 1, 2, and 1 which match the degrees.

The functional equation is simple algebra plus the fact that the dimension is 1, the Euler characteristic is 0, and $\alpha\beta = q$:

$$Z(E, \frac{1}{qT}) = \frac{(1 - \frac{\alpha}{qT})\beta T (1 - \frac{\beta}{qT})\alpha T}{(1 - \frac{1}{qT})\alpha\beta T (1 - \frac{q}{qT})T} = Z(E,T).$$

For the Riemann hypothesis, notice that for $\frac{m}{n} \in \mathbb{Q}$

$$\det(\frac{m}{n} - \phi_l) = \frac{\det(m - n\phi_l)}{n^2} = \frac{\deg(m - n\phi_l)}{n^2} \geq 0$$

Therefore the characteristic polynomial $T^2 - aT + q$ is non-negative on $\mathbb{Q}$, and hence it has a pair of complex conjugate roots or a double real root. In either case, $\alpha = \overline{\beta}$ and hence $|\alpha| = |\beta| = \sqrt{q}$. We conclude

Theorem 7. *The Weil conjectures are true for elliptic curves.*

3. Complex Analogues

The above argument with the Tate module looks a lot like a cohomological argument. Any reasonable "topological" cohomology theory should have the first cohomology group of an elliptic curve be of rank 2, with rank one degree zero and two cohomology groups. The Frobenius has degree q, so on the top cohomology it should be multiplication by q, while on the degree zero cohomology it should be the identity. This accounts for the $1 - qT$ and $1 - T$ terms. The numerator is related to the characteristic polynomial of ϕ acting on the Tate module.

Of course, a topological cohomology theory like singular cohomology doesn't work on an elliptic curve over $\mathbb{F}_q$, and coherent sheaf cohomology doesn't deal with cohomology with constant coefficients, so there are serious obstacles to this approach. Ultimately, the étale cohomology theory developed by Grothendieck succeeded in this quest. A big piece of evidence that suggested this would work was an analogue in complex geometry proven by Serre in a letter to Weil [2]. We will prove this analogue of the Weil conjectures.

Let V be a complex projective variety of dimension n. We will abuse notation and also use V to denote its complex points. Suppose f is an endomorphism, E a hyperplane class, and $q > 0$ an integer such that $f^{-1}(E)$ is algebraically equivalent to qE. The endomorphism f is supposed to play the role of Frobenius. Recall that the Lefschetz number of an endomorphism g is defined to be

$$L(g) = \sum_{j=0}^{2n} (-1)^j \operatorname{tr}\left(g^* | H^j(V, \mathbb{C})\right)$$

If g has non-degenerate fixed points, this counts the number of fixed points.[1] If there was a meaningful way to apply this to the q-power Frobenius, this would be the number of $\mathbb{F}_q$ points. This leads us to define the Lefschetz zeta function as a complex version of the local zeta function by

$$Z_f(V, T) := \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} T^m\right)$$

We will prove the analogue of the Weil conjectures for the Lefschetz zeta function. We define $P_j(T) = \det(1 - f^* T | H^j(V, \mathbb{C}))$, and will use f_j to denote the endomorphism of $H^j(V, \mathbb{C})$ induced by f.

Proposition 8. *The Lefschetz zeta function is a rational of the form*

$$Z_f(V, T) = \prod_{j=0}^{2n} P_j(T)^{(-1)^{j+1}}$$

where $P_j(T)$ has integer coefficients and its degree equals the jth Betti number of V.

[1] As g is holomorphic, all of the signs are positive.

Proof. We calculate using the definitions that

$$\begin{aligned}
\log Z_f(V,t) = \sum_{m=1}^{\infty} \frac{L(f^m)}{m} T^m &= \sum_{m=1}^{\infty}\sum_{j=0}^{2n} (-1)^j \operatorname{tr}\left(f_j^m | H^j(X,\mathbb{C})\right) \frac{T^m}{m} \\
&= \sum_{j=0}^{2n} (-1)^j \operatorname{tr}\left(\sum_{m=1}^{\infty} \frac{f_j^m T^m}{m}\right) \\
&= \sum_{j=0}^{2n} (-1)^j \operatorname{tr}\left(\log\left(\frac{1}{1-f_jT}\right)\right) \\
&= \sum_{j=0}^{2n} (-1)^{j+1} \log\det(1-f_jT)
\end{aligned}$$

Using cohomology with integer coefficients it is clear $P_j(T)$ has integer coefficients. The statement about Betti numbers is immediate. ☐

Next we analyze the eigenvalues of f_k on $H^k(V,\mathbb{C})$. We do this by using Hodge theory to decompose the cohomology and analyze the intersection pairing on the pieces. All of this material may be found in Part II of Voisin [4]. Recall the Hodge decomposition says that $H^k(V,\mathbb{C})$ decomposes as

$$H^k(V,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(V). \tag{1}$$

Remember that elements of $H^{p,q}(V)$ can be represented by equivalence classes of closed differential forms of type (p,q), which makes it clear that $\overline{H^{p,q}(V)} = H^{q,p}(V)$. Furthermore, the cup product is given by the wedge product and Poincaré duality identifies $H^{p,q}(V)$ with $H^{n-q,n-p}(V)$. Note that the class $[\omega]$ of the hyperplane section is of type $(1,1)$ since it is a Kähler class. Then define the Lefschetz operator

$$L : H^k(V) \to H^{k+2}(V)$$

to be given by cupping with $[\omega]$. This obviously restricts to $L : H^{p,q}(V,\mathbb{C}) \to H^{p+1,q+1}(V,\mathbb{C})$.

The Hard Lefschetz theorem says that there is an isomorphism

$$L^{n-(p+q)} : H^{p,q}(V) \simeq H^{n-q,n-p}(V)$$

In particular, this implies L is injective, so $H^{p,q}$ decomposes as a sum of the image of $H^{p-1,q-1}(V)$ under L and a "primitive" complement. More formally, a class $\alpha \in H^{p,q}(V)$ is primitive if $L^{n-(p+q)+1}\alpha = 0$. Denoting the subspace of primitive classes by $H^{p,q}_{\text{prim}}(V)$, there is a Lefschetz decomposition

$$H^{p,q}(V) = \bigoplus_{0 \le 2r \le p+q} L^r H^{p-r,q-r}_{\text{prim}}(V) \tag{2}$$

Remark 9. Using Poincaré duality which identifies $H^{p,q}(V)$ with $H^{n-q,n-p}(V)$, we can define the dual operator $\Lambda : H^{p,q}(V) \to H^{p-1,q-1}$ of L. A calculation shows that $[L,\Lambda] = (k-n)\,\mathrm{Id}$ on $H^k(V)$, so the total cohomology of V is a representation of $\mathfrak{sl}_2$ with raising and lowering operators L and Λ. The lowest weight spaces of this representation are the primitive cohomology.

To prove the analogues of the functional equation and the Riemann hypothesis for $Z_f(V,T)$, we need to understand the eigenvalues of $f^* : H^*(V) \to H^*(V)$. It is more convenient to work instead with $g_k := q^{-k/2} f^*|_{H^k(V)}$. We observe:

- g_k respects the Hodge decomposition (1) because f and g are algebraic (and holomorphic).
- $f^*[\omega] = q[\omega]$ because $f^{-1}(E)$ is equivalent to qE, and hence $g_2([\omega]) = [\omega]$.

- The g_k respect the cup product: if $\alpha \in H^i(V)$ and $\beta \in H^j(V)$, then
$$g_{i+j}(\alpha \cup \beta) = q^{-(i+j)/2} f^*(\alpha \cup \beta) = q^{-i/2} f^*(\alpha) \cup q^{-j/2} f^*(\beta) = g_i(\alpha) \cup g_j(\beta).$$
- The g_k respect the Lefschetz decomposition (2) since they respect the cup product and fix $[\omega]$.

The cup product defines a sesquilinear form on all cohomology groups, not just the middle one: for $[\alpha], [\beta] \in H^k(V, \mathbb{C})$, define
$$H_k([\alpha], [\beta]) = i^k \int_V \alpha \wedge \overline{\beta} \wedge \omega^{n-k}.$$
The g_k respect this form: since g_2 fixes $[\omega]$ and $[\omega]^n$ is a non-zero element in the one dimensional $H^{2n}(V, \mathbb{C})$, g_{2n} is the identity map on top cohomology. Therefore
$$\begin{aligned} H_k(g_k[\alpha], g_k[\beta]) &= i^k \int_V g_k\alpha \wedge g_k\overline{\beta} \wedge \omega^{n-k} \\ &= i^k \int_V g_k\alpha \wedge g_k\overline{\beta} \wedge g_{2n-2k}\omega^{n-k} \\ &= i^k \int_V g_{2n}(\alpha \wedge \overline{\beta} \wedge \omega^{n-k}) \\ &= i^k \int_V \alpha \wedge \overline{\beta} \wedge \omega^{n-k} = H_k([\alpha], [\beta]). \end{aligned}$$

We now need one further fact from Hodge theory, which is part of the technical content of the Hodge index theorem: the Hodge decomposition and the Lefschetz decomposition are orthogonal decompositions with respect to H_k, and is definite on $L^r H^{p-r,q-r}_{\text{prim}}(V)$ (see [4, II.6.3.2]).

Now let v be an eigenvector for g_k. As g_k respects the Hodge and Lefschetz decomposition, it lies in a particular $L^r H^{p-r,q-r}_{\text{prim}}(V)$ on which H_k is definite. Therefore the standard calculation from linear algebra shows that
$$H_k(v, v) = H_k(g_k v, g_k v) = H_k(\lambda v, \lambda v) = |\lambda|^2 H_k(v, v).$$
But as H_k is definite, we can cancel and conclude $|\lambda| = 1$. In terms of the eigenvalues of f_k on $H^k(V, \mathbb{C})$, this implies all of the eigenvalues have absolute value $|q|^{k/2}$. Hence all the roots of $P_k(T) = \det(1 - f_k T)$ have absolute value $|q|^{-k/2}$. This is exactly the analogue of the Riemann hypothesis.

Remark 10. This argument can be phrased in terms of correspondences on the cohomology and the positivity of a certain trace map: see Theorem 2 of Serre [2]. This is an analogue of a formula of Castelnuovo that is used in an algebraic proof of the Weil conjectures for curves. Exactly the same argument uses the complex analogue to deduce the Riemann hypothesis for the Lefschetz zeta function.

Finally, the functional equation follows easily using Poincaré duality. In particular, duality shows that the set of roots of $P_k(T)$ is in bijection with the set of roots of $P_{2n-k}(T)$ under the map $\alpha \mapsto \frac{q^n}{\alpha}$.

Remark 11. Serre's argument suggests that constructing a "topological" cohomology theory in characteristic p would provide an effective attack on the Weil conjectures. Grothendieck developed the theory of étale cohomology as an answer to this challenge. This succeeded in proving all of the Weil conjectures except for the Riemann hypothesis. Grothendieck formulated his standard conjectures to give analogues of the Hodge theory used in order to mimic the above argument. They remain an open problem. Deligne used a slightly different approach, proving the Weil conjectures (and also Grothendieck's version of the hard Lefschetz theorem).

References

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