Projective curvature and integral invariants

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Abstract. In this paper, an extension of all Lie group actions on $\mathbf{R}^2$ to coordinates defined by potentials is given. This provides a new solution to the equivalence problems of curves under the projective group and two of its subgroups. The potentials correspond to integrals of higher and higher order producing an infinite number of independent integral invariants. Applications to computer vision are discussed.

Keywords: Lie group, prolongation, differential invariant, projective curvature, equivalence, potential, integral invariant.

Mathematics Subject Classifications (1991): 53A55

1. Introduction

The projective group plays an important role in computer vision (for example [1]). This group describes all possible viewing transformations on planar objects. The fundamental invariant of the projective group is the projective curvature which characterises curves under a projective transformation. However it depends on seventh order derivatives so is very sensitive to noise and of little practical use. There are three subgroups in the projective group, the Euclidean group E(2), special affine group SA(2) and full affine group A(2) which are important in planar object recognition (see [2]). The fundamental invariants of these groups also depend on derivatives so are sensitive to noise.

It is shown in [3] that the critical points of the special affine curvature are projectively invariant and characterise the singularities in the projective curvature. Also inflection points are projectively invariant. These provide invariant points that can be used to match curves related by a projective transformation. But inflection points correspond to zero Euclidean curvature which depends on second order derivatives and the special affine curvature critical points depend on fourth order derivatives. These are sensitive to noise, so to effectively utilise them to match curves (under a projective transformation) requires an extensive amount of smoothing and tweaking of the projective group parameters. The tweaking is needed to correct the error introduced after smoothing (which distorts the curves), see [3] for details on this.

The root of the problem lies in the derivation of invariants of the projective group and its subgroups. The traditional approach uses derivatives as coordinates to prolong the group action thus the resulting invariants will depend on derivatives (see [4] and [5]). This paper looks at using potentials as coordinates to prolong the group action, thus the resulting invariants will depend on integrals rather than derivatives so will not be as sensitive to noise and will not require the same degree of smoothing.

In section 2.1 it is shown how a general Lie group action on $\mathbf{R}^2$ can be extended to act on potentials and for the case of the full affine group explicit formulas are derived for the extended group action. Using the regularization approach a full affine integral invariant is derived which is used to solve the equivalence problem of curves under the affine group. Furthermore an infinite number of independent integral invariants can be derived.

But as is shown in section 2.2.1, the analytic formulas for the extended action of the projective group on the potentials in section 2.1 (and thus the formulas for the integral invariants of the projective group) cannot be explicitly determined.

In section 2.4 and 2.5 a new formulation of potentials is given which enables explicit formulas for the integral invariants of a subgroup of $PGL(3)$ acting on $\mathbf{R}^2$ (with practical significance, see chapter 3 of [3]) and the projective group to be determined. For the projective group this involves incorporating a differential invariant (involving fifth order derivatives) of the special affine group. But in practice it is the potentials in section 2.1 that are needed to be used as they do not depend on any derivatives. The problem here is that no analytical formula for the resulting integral invariants can be found.

However in section 2.6 it is shown that in principle for any point on a fixed curve, the values of the fully integral projective invariants can be found numerically. A simple example is used to demonstrate the idea.

2. Integral invariants

The groups $E(2)$, $SA(2)$, $A(2)$ and $PGL(3)$ (acting on $\mathbf{R}^2$) which are fundamental in planar object recognition have invariants that depend on high order derivatives so are sensitive to noise. The problem is that derivatives are used in the derivation of the invariants. In this section potentials are used to extend the group action thus the resulting invariants will depend on integrals rather than derivatives so will not be as sensitive to noise. The term integral invariant is used to denote an invariant which requires integration of a curve to evaluate at each point.

2.1. Extending a Lie group action on $\mathbf{R}^2$ to potentials

Consider a Lie group action G (of dimension r) on $\mathbf{R}^2$ defined by

$$(x,u) \longmapsto g\cdot(x,u) = (\bar{x},\bar{u}),\ \ g\in G.$$

The conventional way of deriving invariants is to prolong the action G to the jet space J^{n-2} where $n > r$ and J^n consists of up to $(n-2)$th order derivatives of u (for example, $J^2 = (x,u,u_x,u_{xx})$). The action of G on J^{n-2} can be considered as the extended group action of G on $\mathbf{R}^n$. The goal here is to extend G to an action on $\mathbf{R}^n$ but to use potentials rather than derivatives. The potentials to be considered will be monomials in x and u (see section 2.4 and 2.5 for examples using more complicated definitions of potentials).

Definition 1. *The **monomial potential** $V^{i,j}$ of order k is given by*

$$V^{i,j}_x = x^i u^j,$$

where $j \neq 0$ and $i+j=k$.

Put

$$z = V^{0,1},\ \ v = V^{1,1},\ \ w = V^{0,2}. \tag{1}$$

Thus z is an order 1 monomial potential and v, w are order 2 monomial potentials. There are $K = \frac{k(k+1)}{2}$ order k monomial potentials,

$$V^{k-1,1}_x = x^{k-1}u,\ V^{k-2,2}_x = x^{k-2}u^2,\ \dots\ ,\ V^{0,k}_x = u^k.$$

For a given curve $u = u(x)$ through (x_0,u_0) which will be viewed as the initial conditions of u,

$$z = \int_{x_0}^{x} u\,dx, \quad v = \int_{x_0}^{x} xu\,dx, \quad w = \int_{x_0}^{x} u^2\,dx \tag{2}$$

Definition 2. *The **potential jet space** J^n_p, $n \geq 2$ is the Euclidean space $\mathbf{R}^{n+4}$ with local coordinates $(x,u,x_0,u_0,V_{(n)})$ where $V_{(n)}$ consists of monomial potentials up to kth order (where k is the largest integer such that $K \leq n$) plus (if $n \neq K$) the potentials*

$$V^{k,1},\ V^{k-1,2},\ \dots\ ,\ V^{k-(n-K)+1,n-K}$$

For $n=1$, $J^1_p = (x,u,x_0,u_0,V^{0,1})$.

For example,

$$J_p^2 = (x, u, x_0, u_0, V^{0,1}, V^{1,1}) \simeq \mathbf{R}^6$$
$$J_p^3 = (x, u, x_0, u_0, V^{0,1}, V^{1,1}, V^{0,2}) \simeq \mathbf{R}^7$$

Proposition 1. *G extends to a group action on $J_p^n \simeq \mathbf{R}^{n+4}$ for $n \geq 1$ by*

$$\begin{aligned}(x, u, x_0, u_0, V_{(n)}) \longmapsto &(g \cdot x, g \cdot u, g \cdot x_0, g \cdot u_0, g \cdot V_{(n)}) \\ &= (\bar{x}, \bar{u}, \bar{x}_0, \bar{u}_0, \bar{V}_{(n)})\end{aligned}$$

where $\bar{V}_{(n)} = (\bar{V}^{0,1}, \bar{V}^{1,1}, \bar{V}^{0,2}, \dots)$ are defined by

$$\bar{V}_{\bar{x}}^{0,1} = \bar{u}, \quad \bar{V}_{\bar{x}}^{1,1} = \bar{x}\bar{u}, \quad \bar{V}_{\bar{x}}^{0,2} = \bar{u}^2, \dots$$

Proof. Using (2), for a curve $u = u(x)$ through (x_0, u_0),

$$\begin{aligned}\bar{z} = g \cdot z &= \int_{\bar{x}_0}^{\bar{x}} \bar{u}\, d\bar{x} \\ &= \int_{x_0}^{x} g \cdot u \frac{d(g \cdot x)}{dx} dx = \int_{x_0}^{x} \bar{u} \frac{d\bar{x}}{dx} dx. \end{aligned} \tag{3}$$

Thus

$$\begin{aligned}h \cdot (g \cdot z) = h \cdot \bar{z} &= \int_{\bar{x}_0}^{\bar{x}} h \cdot \bar{u}\, \frac{d(h \cdot \bar{x})}{d\bar{x}} d\bar{x} \\ &= \int_{x_0}^{x} (h \cdot g) \cdot u\, \frac{d((h \cdot g) \cdot x)}{dx} dx \\ &= (h \cdot g) \cdot z\end{aligned}$$

(since have a group action on (x, u)), similarly for all the higher order monomial potentials. This combined with the identity element and inverse (determined from the group action on (x, u)) completes the proof. □

2.2. The Full Affine Group

Consider the full affine group action on $\mathbf{R}^2$ given by,

$$(x, u) \longmapsto (ax + bu + c, dx + eu + f), \quad \det \begin{pmatrix} a & b \\ d & e \end{pmatrix} \neq 0.$$

The group action of $A(2)$ on z is given by

$$\begin{aligned}\frac{d\bar{z}}{dx} &= (dx + eu + f)(a + bu_x)\\ &= (ae - bd)u + (afx + \frac{1}{2}adx^2 + bdxu + \frac{1}{2}beu^2 + bfu)'\end{aligned}$$

and so

$$\begin{aligned}\bar{z} &= (ae - bd)z + \frac{1}{2}ad(x^2 - x_0^2) + bd(xu - x_0u_0)\\ &+ \frac{1}{2}be(u^2 - u_0^2) + af(x - x_0) + bf(u - u_0)\end{aligned}$$

(using (3)). Similarly,

$$\begin{aligned}\bar{v} &= c(ae - bd)z + a(ae - bd)v + \frac{1}{2}b(ae - bd)w\\ &+ \frac{1}{2}b(bf + ec)(u^2 - u_0^2) + \frac{1}{2}a(dc + af)(x^2 - x_0^2) + \frac{1}{3}b^2e(u^3 - u_0^3)\\ &+ bcd(xu - x_0u_0) + bcf(u - u_0) + \frac{1}{2}abe(xu^2 - x_0u_0^2)\\ &+ abf(xu - x_0u_0) + acf(x - x_0) + \frac{1}{3}a^2d(x^3 - x_0^3)\\ &+ abd(x^2u - x_0^2u_0) + \frac{1}{2}b^2d(xu^2 - x_0u_0^2),\end{aligned}$$

$$\begin{aligned}\bar{w} &= 2f(ae - bd)z + 2d(ae - bd)v + e(ae - bd)w + bfe(u^2 - u_0^2)\\ &+ afd(x^2 - x_0^2) + bd^2(x^2u - x_0^2u_0) + bde(xu^2 - x_0u_0^2)\\ &+ 2bfd(xu - x_0u_0) + af^2(x - x_0) + \frac{1}{3}be^2(u^3 - u_0^3)\\ &+ \frac{1}{3}ad^2(x^3 - x_0^3) + bf^2(u - u_0),\end{aligned}$$

where $(x_0, u_0) = (x_0, u(x_0))$ are the initial conditions of the curve $u = u(x) \in \mathbf{R}^2$. Formulas for $\bar{V}^{2,1}$, $\bar{V}^{1,2}$, $\bar{V}^{0,3}$, $\ldots$ can also be found in a similar way where

$$\bar{V}^{2,1}_{\bar{x}} = \bar{x}^2\bar{u},\ \bar{V}^{1,2}_{\bar{x}} = \bar{x}\bar{u}^2,\ \bar{V}^{0,3}_{\bar{x}} = \bar{u}^3,\ \ldots$$

(see section 2.1). The regularization approach allows the parameters a, b, c, d, e and f to be normalized by setting

$$(\bar{x}, \bar{u}, \bar{z}, \bar{v}, \bar{x}_0, \bar{u}_0) = (0, 0, 0, 0, 1, 1). \tag{4}$$

This gives a map $\rho : Z = (x, u, x_0, u_0, z, v) \longmapsto A(2)$ (satisfying $\rho(g \cdot Z) = \rho(Z) \cdot g^{-1}$) defined by

$$\rho(Z) = \begin{pmatrix} a(Z) & b(Z) & c(Z) \\ d(Z) & e(Z) & f(Z) \\ 0 & 0 & 1 \end{pmatrix} \in A(2)$$

which is called a right moving frame ($a(Z), \ldots, f(Z)$ are the unique solution to (4)). Substituting $a = a(Z), \ldots, f = f(Z)$ into $\bar{w}$ gives the curvature

$$\kappa = \frac{1}{3}\Big(-3(x-x_0)w + 6(u-u_0)v + 4z^2 - 2(2xu-2x_0u_0+ux_0-xu_0)z + uu_0(x-x_0)^2\Big)\Big/\Big((x-x_0)(u+u_0)-2z)^2\Big). \tag{5}$$

The existence of a moving frame implies the group action is free and regular. Thus, following Olver ([4], [5]) the classifying curve (or signature) can be defined as $\mathcal{C} = (\kappa, \kappa_2)$, where κ_2 is found by substituting the moving frame into $\bar{V}^{2,1}$ and $\mathcal{C}$ characterises all curves up to an affine transformation (a consequence of Cartan's equivalence theorems, see [5], [7], [8]). In practice given a curve $u = u(x)$ through (x_0, u_0) this curvature can be evaluated for each point x by using (2). Note that if two curves C and $\bar{C}$ are related by an affine transformation a point $(\bar{x}_0, \bar{u}_0) \in \bar{C}$ must be identified with the point $(x_0, u_0) \in C$ to evaluate the curvature $\bar{\kappa}$. Geometrically the curvature is a combination of point values and areas under the graphs of u, xu and u^2. For example at the point $(x(t_1), y(t_1))$ on the curve $x = x(t), u = u(t)$, $z = \int_{t_0}^{t_1} u(t)x'(t)\, dt$ which is area A - area B in figure 1.

Note that this integral invariant is different from the traditional moment invariants, see [6] and [10] for example, as the moment invariants are global invariants where this invariant is semi-local since by varying (x_0, u_0) it can be defined on any segment of the curve. Also another type of integral invariant has been looked at in [9] but this involves integrating with respect to the invariant special affine arclength.

The numerator and denominator of the integral curvature κ above transform by

$$\text{numer}(\kappa) \longmapsto (ae - bd)^2 \text{numer}(\kappa)$$
$$\text{denom}(\kappa) \longmapsto (ae - bd)^2 \text{denom}(\kappa)$$

Thus, in the case of the special affine group where $ae - bd = 1$, the numerator and denominator of κ are separately invariant.

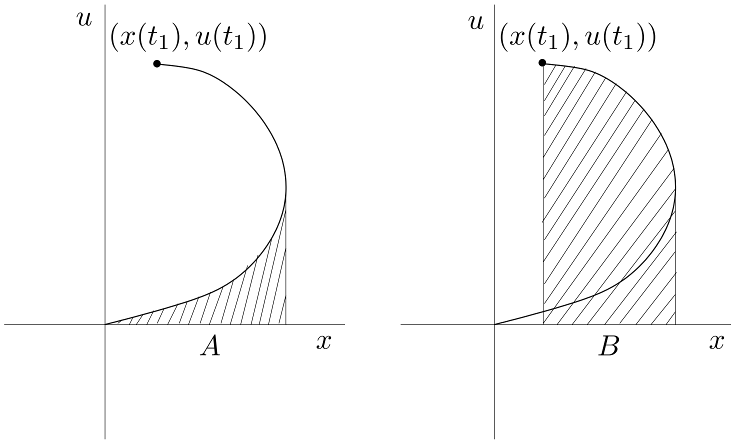

Figure 1. Geometric interpretation of the coordinate z in the integral curvature.

2.2.1. *Non-exact extensions*

Consider the (local) projective group action of $PGL(3)$ $(= SL(3))$ on $\mathbf{R}^2$ given by

$$(x, u) \longmapsto \left(\frac{ax + bu + c}{gx + hu + i}, \frac{dx + eu + f}{gx + hu + i} \right), \quad \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1 \qquad (6)$$

Following the same process as with the full affine group this gives,

$$\frac{d\bar{z}}{dx} = \frac{1}{2} \left(\frac{(ax + bu + c)(dx + eu + f)}{(gx + hu + i)^2} \right)' + \frac{1}{2}\frac{1}{h} \left(\frac{(ea - bd)x - (bf - ce)}{gx + hu + i} \right)' - \frac{1}{2}\frac{1}{h}\frac{1}{(gx + hu + i)^2}.$$

Hence in $\bar{z}$ there will be a term

$$\int_{x_0}^{x} \frac{1}{(gx + hu + i)^2} \, dx$$

which cannot be determined (similarly for $\bar{v}$ and $\bar{w}$). In section 2.4 and 2.5 a new formulation of potentials is given for a subgroup of $PGL(3)$ and the projective group so that the equations for $\frac{d\bar{z}}{dx}$, $\frac{d\bar{v}}{dx}$ and $\frac{d\bar{w}}{dx}$ are exact. (For the projective group this involves introducing a differential invariant of the special affine group).

2.3. An Example

Consider the two convex curves $(X(t), Y(t))$ and $(\bar{X}(t), \bar{Y}(t))$ where

$$X(t) = \sin(t) - \frac{1}{5}\cos(t)^2, \quad Y(t) = \frac{1}{2}\sin(t) - \cos(t)$$

and $(\bar{X}(t), \bar{Y}(t))$ is related to $(X(t), Y(t))$ by a full affine transformation. They are parametrised by t from 0 to 2π and are each translated by mapping a common point in each curve to the origin so that $c = 0$ and $f = 0$.

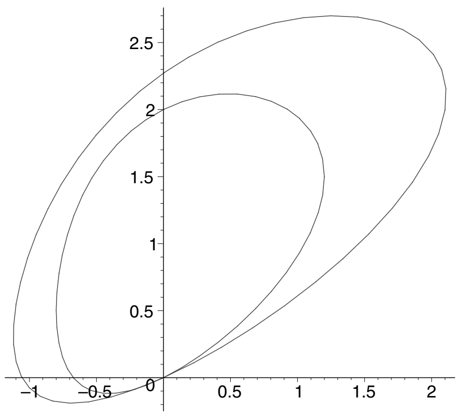

Figure 2. Two curves related by a full affine transformation with $c = 0$ and $f = 0$.

Since $(x_0, u_0) = (\bar{x}_0, \bar{u}_0) = (0, 0)$ the above curvature takes the simpler form

$$\kappa = \frac{1}{3}\frac{-3xw + 6uv + 4z^2 - 4xuz}{(xu - 2z)^2}$$

Consider

$$\kappa_1 = xu - 2z.$$

and so

$$\kappa_1(2\pi) = -2\int_0^{2\pi} u(t)x'(t)\,dt$$

in parametrised form, where $x(0) = 0$ and $u(0) = 0$. Note that if a translation is performed on $x(t)$ and $u(t)$

$$\tilde{x}(t) = x(t) + c$$
$$\tilde{u}(t) = u(t) + f$$

then for arbitrary t_0,

$$\begin{aligned}\int_{t_0}^{t_0+2\pi} \tilde{u}(t)\tilde{x}'(t)\,dt &= \int_{t_0}^{t_0+2\pi} (u(t)+f)x'(t)\,dt \\ &= \int_{t_0}^{t_0+2\pi} u(t)x'(t)\,dt + f\int_{t_0}^{t_0+2\pi} x'(t)\,dt \\ &= \int_{0}^{2\pi} u(t)x'(t)\,dt.\end{aligned}$$

Hence the value of $\kappa_1(2\pi)$ is independent of translation of the curve. Under a general affine transformation, $\bar{\kappa}_1(2\pi) = (ae-bd)\kappa_1(2\pi)$, so the values of $\bar{\kappa}_1(2\pi)$ and $\kappa_1(2\pi)$ can be used to determine $(ae-bd)$ independent of translation along the curve thus the full affine transformation will become special affine.

For the curves in figure 2, the factor $(ae-bd)$ is found by the above method and κ_1 and $\frac{1}{ae-bd}\bar{\kappa}_1$ are both plotted against the parameter t. Figure 3 shows that the two graphs are identical.

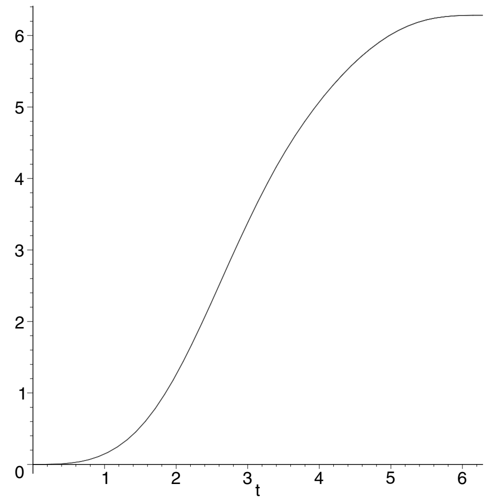

Figure 3. Plotting κ_1 and $\frac{1}{ae-bd}\bar{\kappa}_1$ vs t.

The integral curvature's κ and $\bar{\kappa}$ are next plotted against t and figure 4 shows they are identical.

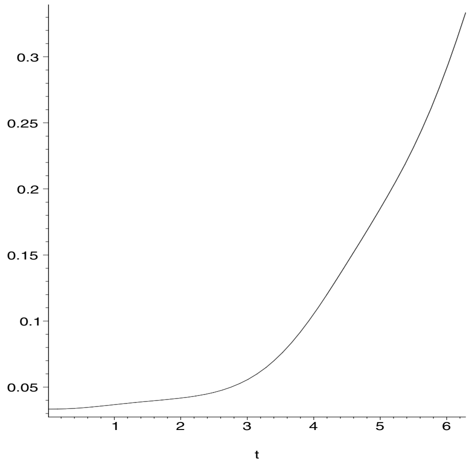

Figure 4. The integral curvatures κ and $\bar{\kappa}$ vs t.

2.4. Subgroup of $PGL(3)$ acting on $\mathbf{R}^2$

Consider the action of the subgroup

$$G = \{ \begin{pmatrix} a & b & 0 \\ 0 & 1 & 0 \\ g & h & i \end{pmatrix} \subset PGL(3)\}$$

on $\mathbf{R}^2$ given by

$$(x, u) \longmapsto (\bar{x}, \bar{u}) = \left(\frac{ax + bu}{gx + hu + i}, \frac{u}{gx + hu + i} \right). \tag{7}$$

This has practical significance (see chapter 3 of [3]). As was shown in section 2.2.1, for the potentials $z_x = u$, $v_x = xu$ and $w_x = u^2$, the equations of $\frac{d\bar{z}}{dx}$, $\frac{d\bar{v}}{dx}$ and $\frac{d\bar{w}}{dx}$ are not exact. Hence for these potentials, analytical expressions for the integral invariants of the action (7) cannot be determined. In this section a new formulation of potentials is given so that the integral invariants can be derived.

New coordinates v and w can be defined by,

$$v_x = \frac{1}{u^2}, \qquad w_x = \frac{x}{u^3}.$$

Similar to section 2.1 it can be shown that the group action (7) can be extended to a group action on v and w. The transformations are given by

$$\bar{v} = aiv + (ah - bg)\frac{x}{u} - bi\frac{1}{u} - (ah - bg)\frac{x_0}{u_0} + bi\frac{1}{u_0}$$

$$\bar{w} = a^2 iw + \frac{3}{2}abiv - \frac{1}{2}abi\frac{x}{u^2} + b((ah - bg)\frac{x}{u} - bi\frac{1}{u}) - \frac{1}{2}a(ah - bg)\frac{x_0^2}{u_0^2}$$
$$+ \frac{1}{2}abi\frac{x_0}{u_0^2} - b(ah - bg)\frac{x_0}{u_0} + b^2 i\frac{1}{u_0} + \frac{1}{2}a(ah - bg)\frac{x^2}{u^2}$$

where $(x_0, u(x_0))$ is the initial condition of the curve $u \in \mathbf{R}^2$ and $u(x_0) \neq 0$.
Also the group action can be extended to the coordinates $(w^{(k)}, k = 1, \ldots, n)$ where $w_x^{(k)} = \frac{x^k}{u^{k+2}}$ for arbitrary n. Thus an infinite number of independent integral invariants can be derived.

Now using the regularization approach the parameters a, b, g, h and i can be normalized by setting

$$(\bar{x}, \bar{u}, \bar{v}, \bar{x}_0, \bar{u}_0) = (0, 1, 0, 1, 1)$$

and then they can be substituted into $\bar{w}$ to give the integral curvature

$$\kappa = -\frac{1}{2}\frac{u_0(x_0u^2v - 2u_0u^2w + 2u_0xuv + x_0x - x^2)}{(u_0uv + x_0 - x)(x_0u - xu_0)}.$$

In similar way to section 2.2 this can be used to solve the equivalence problem of curves under the group action (7). For a given curve $u = u(x)$ through the point (x_0, u_0) this integral curvature can be evaluated at each point x by using,

$$v = \int_{x_0}^{x} \frac{1}{u^2}\, dx, \quad w = \int_{x_0}^{x} \frac{x}{u^3}\, dx.$$

2.5. The Projective Group

In this section potentials involving an invariant of the special affine group are used so that integral invariants for the projective group can be derived. New coordinates I, I_x, I_u, I_{xx}, I_{xu}, I_{uu}, can be defined by,

$$\frac{d}{ds}I = \kappa_s^{\mathbf{SA}}, \quad \frac{d}{ds}I_x = x\kappa_s^{\mathbf{SA}}, \quad \frac{d}{ds}I_u = u\kappa_s^{\mathbf{SA}},$$
$$\frac{d}{ds}I_{xx} = x^2\kappa_s^{\mathbf{SA}}, \quad \frac{d}{ds}I_{xu} = xu\kappa_s^{\mathbf{SA}}, \quad \frac{d}{ds}I_{uu} = u^2\kappa_s^{\mathbf{SA}}$$

where s is the special affine arclength ($ds = u_{xx}^{\frac{1}{3}}dx$) and

$$\kappa_s^{\mathbf{SA}} = \frac{1}{3}\frac{9u_{xx}^2u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3}{u_{xx}^4}$$

The projective group can be extended to these coordinates (in a similar way to section 2.2 and 2.4) where

$$\begin{aligned}
\bar{I} &= g^2 I_{xx} + 2gh I_{xu} + h^2 I_{uu} + 2gi I_x + 2hi I_u + i^2 I \\
\bar{I}_{\bar{x}} &= ag I_{xx} + (ah + bg) I_{xu} + bh I_{uu} + (ai + gc) I_x + (bi + hc) I_u + ci I \\
\bar{I}_{\bar{u}} &= dg I_{xx} + (dh + ge) I_{xu} + eh I_{uu} + (di + gf) I_x + (ei + hf) I_u + fi I \\
\bar{I}_{\bar{x}\bar{x}} &= a^2 I_{xx} + 2ab I_{xu} + b^2 I_{uu} + 2ac I_x + 2bc I_u + c^2 I \\
\bar{I}_{\bar{x}\bar{u}} &= ad I_{xx} + (ea + bd) I_{xu} + be I_{uu} + (fa + dc) I_x + (fb + ec) I_u + cf I \\
\bar{I}_{\bar{u}\bar{u}} &= d^2 I_{xx} + 2de I_{xu} + e^2 I_{uu} + 2df I_x + 2ef I_u + f^2 I
\end{aligned}$$

The parameters may be normalized by setting

$$(\bar{x}, \bar{u}, \bar{I}, \bar{I}_{\bar{x}}, \bar{I}_{\bar{u}}, \bar{I}_{\bar{x}\bar{u}}, \bar{x}_0, \bar{u}_0) = (0, 0, 1, 0, 0, 0, 0, 1)$$

and substituting into $\bar{I}_{\bar{x}\bar{x}} \cdot \bar{I}_{\bar{u}\bar{u}}$ gives the invariant

$$\kappa_I^{\text{proj}} = -I_{xx} I_u^2 + I_{xx} I_{uu} I + 2 I_{xu} I_x I_u - I_{uu} I_x^2 - I_{xu}^2 I.$$

which can be used to solve the equivalence problem of curves under the projective group (similar to section 2.2).

Note that an alternative way of deriving integral invariants is to use a subgroup to solve for the larger group with potentials (like one can do with differential invariants, see example in appendix). For complicated group actions this has the advantage of simpler computations.

This invariant can be evaluated by using,

$$\begin{aligned}
I &= \int_0^s \kappa_s^{\mathbf{SA}} ds, \quad I_x = \int_0^s x \kappa_s^{\mathbf{SA}} ds, \quad I_u = \int_0^s u \kappa_s^{\mathbf{SA}} ds, \\
I_{xx} &= \int_0^s x^2 \kappa_s^{\mathbf{SA}} ds, \quad I_{xu} = \int_0^s x u \kappa_s^{\mathbf{SA}} ds, \quad I_{uu} = \int_0^s u^2 \kappa_s^{\mathbf{SA}} ds
\end{aligned}$$

where in each integral s varies from a common starting position which is designated to be $s = 0$ corresponding to some point (x_0, u_0) (the initial condition of the curve $u \in \mathbf{R}^2$). Note that using integration by parts the coordinates can be written in the following form,

$$\begin{aligned}
&I = \kappa^{\mathbf{SA}} - \kappa_0^{\mathbf{SA}}, \quad I_x = x\kappa^{\mathbf{SA}} - x_0 \kappa_0^{\mathbf{SA}} - \int_{x_0}^x \kappa^{\mathbf{SA}}\, dx, \\
&I_u = u\kappa^{\mathbf{SA}} - u_0 \kappa_0^{\mathbf{SA}} - \int_{u_0}^u \kappa^{\mathbf{SA}}\, du, \quad I_{xx} = x^2 \kappa^{\mathbf{SA}} - x_0^2 \kappa_0^{\mathbf{SA}} - 2\int_{x_0}^x x\kappa^{\mathbf{SA}}\, dx, \\
&I_{xu} = xu\kappa^{\mathbf{SA}} - x_0 u_0 \kappa_0^{\mathbf{SA}} - \int_{x_0}^x u\kappa^{\mathbf{SA}}\, dx - \int_{u_0}^u x\kappa^{\mathbf{SA}}\, du, \\
&I_{uu} = u^2 \kappa^{\mathbf{SA}} - u_0^2 \kappa_0^{\mathbf{SA}} - 2\int_{u_0}^u u\kappa^{\mathbf{SA}}\, du
\end{aligned}$$

Also since

$$\kappa^{\mathbf{SA}} = \frac{d^2}{dx^2}(u_{xx}^{-\frac{2}{3}}) = \frac{d^2}{du^2}(x_{uu}^{-\frac{2}{3}}),$$

κ_I^{proj} can be written in terms of $\kappa^{\mathbf{SA}}$, $\int_0^s ds$ and up to third order derivatives of u.

Consider two convex curves $(X(t), Y(t))$ (given in section 2.2) and $(\bar{X}(t), \bar{Y}(t))$ which is related to $(X(t), Y(t))$ by a projective transformation. They are each translated by mapping a common point in each curve to the origin so that $c = 0$, $f = 0$, see figure 5.

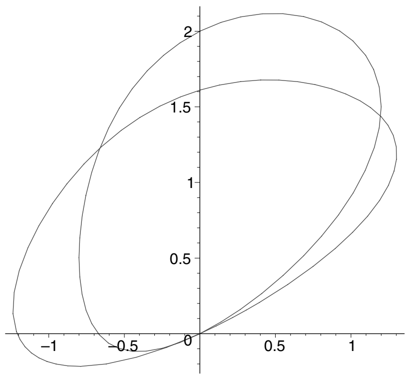

Figure 5. Two curves related by a projective transformation with $c = 0$ and $f = 0$.

The integral curvature's κ_I^{proj} and $\bar{\kappa}_I^{\text{proj}}$ are each plotted against t and figure 6 shows they are identical.

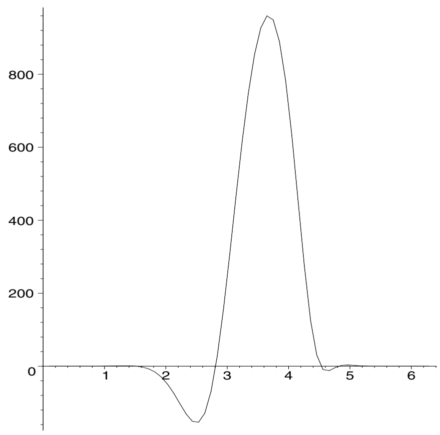

Figure 6. The integral curvatures κ_I^{proj} and $\bar{\kappa}_I^{\text{proj}}$ vs t.

2.6. Evaluating integral invariants when the analytical formula is unknown.

The potentials used for the projective group in section 2.5 involve a differential invariant of the special affine group with fifth order derivatives. In practice it is the potentials in section 2.1 that are needed to be used as they do not depend on any derivatives. The problem here is that the equations for $\frac{d\bar{z}}{dx}$, $\frac{d\bar{v}}{dx}$ and $\frac{d\bar{w}}{dx}$ are not exact. In this section it is shown that in principle the projective integral invariants can still be evaluated at each point on a curve $u \in \mathbf{R}^2$. To demonstrate the idea consider the following group action on $\mathbf{R}^2$

$$(x,u) \longmapsto \left(\frac{ax}{gx+1}, \frac{u}{gx+1}\right) \tag{8}$$

For $z_x = u$ this gives

$$\begin{aligned} \frac{d\bar{z}}{dx} &= \frac{au}{(gx+1)^3} \\ &= \frac{1}{2}a\left(\frac{xu}{(gx+1)^2}\right)' - \frac{1}{2}\frac{a}{g}\left(\frac{u}{gx+1}\right)' + \frac{1}{2}\frac{a}{g}\frac{u_x}{(gx+1)^2} \end{aligned}$$

and so

$$\begin{aligned} \bar{z} &= \int_0^x \frac{d\bar{z}}{dx}\, dx \\ &= \frac{1}{2}\frac{axu}{(gx+1)^2} - \frac{1}{2}\frac{a}{g}\frac{u}{gx+1} + \frac{1}{2}\frac{a}{g}\int_0^x \frac{u_x}{(gx+1)^2}\, dx. \end{aligned}$$

Note that $\int_0^x \frac{u_x}{(gx+1)^2}\, dx$ can be written as $\int_0^u \frac{1}{(gx+1)^2}\, du$. The parameters a and g can be normalized by setting $(\bar{x},\bar{u}) = (1,1)$ which gives

$$a = \frac{u}{x}, \quad g = \frac{u-1}{x}.$$

But they cannot be substituted into $\bar{z}$ to get the analytical expression for the integral curvature as there is an integral in $\bar{z}$ (that cannot be determined) which depends on g. However for any point on a **known** curve $u = u(x)$ the integral curvature can be evaluated.

Consider two curves $u(x) = x^8 - x^2 + \frac{3}{4}x$ (defined on $[0,1]$) and $\bar{u} = \bar{u}(\bar{x})$ which is related to $u(x)$ by the transformation (8), see figure 7.

For $u(x_0)$, the integral curvature κ at a point x_1 is given by

$$\kappa(x_1) = \left[a\int_0^{x_1} \frac{(x^8 - x^2 + \frac{3}{4}x)}{(gx+1)^3}\, dx\right]\Bigg|_{\left(a=\frac{u(x_1)}{x_1},\ g=\frac{u(x_1)-1}{x_1}\right)}$$

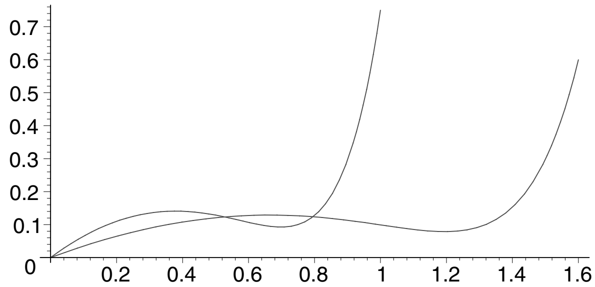

Figure 7. Two curves related by the transformation (8).

where $g = \frac{u(x_1)-1}{x_1}$ is substituted **after** integration. For $\bar{u}(\bar{x})$, the integral curvature $\bar{\kappa}$ at the corresponding point $\bar{x}_1$ is given by

$$\bar{\kappa}(\bar{x}_1) = \left[\bar{a} \int_0^{\bar{x}_1} \frac{\bar{u}(\bar{x})}{(\bar{g}\bar{x}+1)^3}\, d\bar{x} \right] \Bigg|_{\left(\bar{a}=\frac{\bar{u}(\bar{x}_1)}{\bar{x}_1},\ \bar{g}=\frac{\bar{u}(\bar{x}_1)-1}{\bar{x}_1}\right)}$$

Note that a and g are different from $\bar{a}$ and $\bar{g}$. After a computation in **MAPLE** $\kappa(x)$ is plotted against $\bar{\kappa}(\bar{x})$ on the interval $[0, 1]$ and as figure 8 shows they are identical.

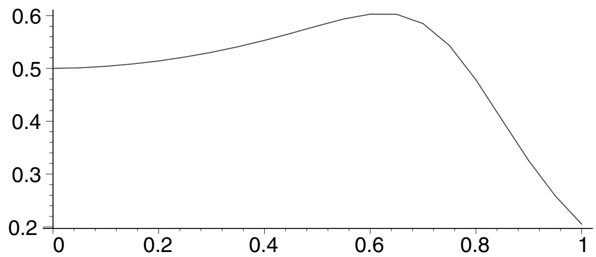

Figure 8. The integral curvatures $\kappa(x)$ and $\bar{\kappa}(\bar{x})$ vs x

For the projective group, as is shown in section 2.2.1

$$\bar{z} = \frac{1}{2}\left[\frac{(ax+bu+c)(dx+eu+f)}{(gx+hu+i)^2}\right]_{x_0}^{x} + \frac{1}{2}\frac{1}{h}\left[\frac{(ae-bd)x-(bf-ce)}{gx+hu+i}\right]_{x_0}^{x} - \frac{1}{2}\frac{1}{h}\int_{x_0}^{x}\frac{1}{(gx+hu+i)^2}\,dx.$$

Similar expressions can be derived for the coordinates v, w, Z, V, W where $v_x = xu$, $w_x = u^2$, $Z_x = xu^2$, $V_x = x^2u$ and $W_x = u^3$. The parameters a, b, c, d, e and f can be normalized directly as they do not occur in any of the integrals. For g and h consider a known curve

$u = u(x)$. The integral in $\bar{z}$ at a point $x = x_n$ can be approximated by summing up all the areas of the trapeziums under the graph $\frac{1}{(gx+hu+i)^2}$ as shown in figure 9.

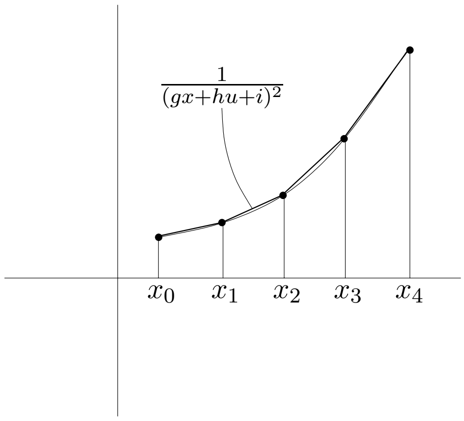

Figure 9. Approximating $\bar{z}$.

This gives

$$\int_{x_0}^{x} \frac{1}{(gx+hu+i)^2}\, dx \simeq \frac{1}{(gx_0+hu_0+i)^2} + 2\sum_{j=1}^{n-1} \frac{1}{(gx_j+hu_j+i)^2} + \frac{1}{(gx_n+hu_n+i)^2}$$

Similarly for $\bar{v}$, $\bar{w}$, $\bar{Z}$, $\bar{V}$ and $\bar{W}$. Thus the normalization equations to solve for g and h will be polynomial equations which for each point x can be solved iteratively. The parameters a, b, c, d, e, f, g, h can be normalized using $\bar{x}, \bar{u}, \bar{x}_0, \bar{u}_0, \bar{z}, \bar{v}, \bar{w}, \bar{Z}$ (eight equations in eight unknowns) and substituted into $\bar{V}$ and $\bar{W}$ to give the evaluation of the projective integral invariants at the point x for the known curve $u(x)$. This will not depend on any derivatives. Hence in principal for a known curve $u(x)$ the projective integral invariants can be evaluated at each point x.

3. Conclusion

The projective curvature (see appendix) depends on seventh order derivatives so is very sensitive to noise and of little practical use in planar object recognition. The differential invariants of $E(2)$, $SA(2)$ and $A(2)$ are also important in object recognition (see [2]) but they are sensitive to noise. The problem is that the group actions are extended to derivatives hence the invariants depend on derivatives.

In this paper it is shown that a general Lie group action on $\mathbf{R}^2$ can be extended to act on potentials which correspond to integrals of x and u and do not depend on derivatives. For the affine group and a subgroup of $GL(3)$ acting on $\mathbf{R}^2$ integral invariants can be explicitly derived. For the projective group new potentials depending on derivatives have to be introduced to get an explicit expression for the integral invariants. But in practice it is the potentials that don't depend on derivatives that are needed to be used. The problem with this is that no analytical expression for the integral invariants can be found. However it is shown in principle (see section 2.6) that at each point x in a known curve $u = u(x)$, the projective integral invariants can be evaluated.

The major future application is to implement the computation of the projective integral invariants of section 2.6 (which do not depend on derivatives) and to incorporate into a practical planar object recognition algorithm.

Another application is to investigate the extension of Lie group actions to potentials for space curves and surfaces in $\mathbf{R}^3$.

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Appendix

A. Alternative derivation of projective curvature

In this section a new derivation of the projective curvature is given. This uses the differential invariants of the special affine group as coordinates to prolong the projective group action. The approach is based on the regularization method described in [5]. It is an example of the general concept of using the simpler equivalence problem based on a subgroup to aid the solution to the more complicated equivalence problem with the larger group thus simplifying computations. This general concept is mentioned in chapter 10 of [8].

Under the projective group, the special affine arclength,

$$ds_{\mathbf{SA}} = u_{xx}^{\frac{1}{3}} dx$$

transforms by

$$d\bar{s}_{\mathbf{SA}} = \frac{ds_{\mathbf{SA}}}{gx + hu + i}.$$

After a calculation using **MAPLE**, it is found that $\kappa^{\mathbf{SA}}$ transforms by

$$\bar{\kappa}^{\mathbf{SA}} = (gx + hu + i)^2 \kappa^{\mathbf{SA}} + 18h(gx + hu + i) u_{xx}^{\frac{1}{3}} - 9\frac{(g + hu_x)^2}{u_{xx}^{\frac{2}{3}}} - 6\frac{(gx + hu + i)(g + hu_x)u_{xxx}}{u_{xx}^{\frac{5}{3}}}.$$

and $\bar{\kappa}^{\mathbf{SA}}_{\bar{s}}$ transforms by

$$\bar{\kappa}^{\mathbf{SA}}_{\bar{s}} = (gx + hu + i)^3 \kappa^{\mathbf{SA}}_s.$$

Similarly, $\bar{\kappa}^{\mathbf{SA}}_{\bar{s}\bar{s}}$ and $\bar{\kappa}^{\mathbf{SA}}_{\bar{s}\bar{s}\bar{s}}$ can be found giving

$$\bar{\kappa}^{\mathbf{SA}}_{\bar{s}\bar{s}} = (gx + hu + i)^4 \kappa^{\mathbf{SA}}_{ss} + 3\frac{(gx + hu + i)^3 (g + hu_x)}{u_{xx}^{\frac{1}{3}}} \kappa^{\mathbf{SA}}_s$$

$$\bar{\kappa}^{\mathbf{SA}}_{\bar{s}\bar{s}\bar{s}} = (gx + hu + i)^3 \Big((gx + hu + i)^2 \kappa^{\mathbf{SA}}_{sss} + 7\frac{(gx + hu + i)(g + hu_x)}{u_{xx}^{\frac{1}{3}}} \kappa^{\mathbf{SA}}_{ss} + \frac{3h(gx + hu + i)u_{xx}^2 + 9(g + hu_x)^2 u_{xx} - (gx + hu + i)(g + hu_x)u_{xxx}}{u_{xx}^{\frac{5}{3}}} \kappa^{\mathbf{SA}}_s \Big)$$

Thus the projective group gives an action on $(\kappa^{\mathbf{SA}}_s, \kappa^{\mathbf{SA}}_{ss}, \kappa^{\mathbf{SA}}_{sss})$ and by setting

$$(\bar{\kappa}^{\mathbf{SA}}_{\bar{s}}, \bar{\kappa}^{\mathbf{SA}}_{\bar{s}\bar{s}}, \bar{\kappa}^{\mathbf{SA}}_{\bar{s}\bar{s}\bar{s}}) = (1, 0, 0),$$

the parameters can be normalized and substituted into $\bar{\kappa}^{\mathbf{SA}}$ which yields

$$\kappa^{\mathrm{proj}} = \frac{-6\kappa_s^{\mathbf{SA}}\kappa_{sss}^{\mathbf{SA}} + 7(\kappa_{ss}^{\mathbf{SA}})^2 + \kappa^{\mathbf{SA}}(\kappa_s^{\mathbf{SA}})^2}{{\kappa_s^{\mathbf{SA}}}^{\frac{8}{3}}}.$$