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SYMMETRIC MATRICES REPRESENTABLE BY WEIGHTED TREES OVER A CANCELLATIVE ABELIAN MONOID *

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Abstract. The classical result that characterizes metrics induced by paths in a labeled tree having positive real edge weights is generalized to allow the edge weights to take values in any cancellative abelian monoid satisfying the additional requirement that $x + x = y + y$ implies $x = y$. This includes the case of arbitrary real-valued edge weights, which applies to distance-hereditary graphs, thus yielding (unique) weighted tree representations for the latter.

Key words. trees, 4-point condition, abelian monoid, distance-hereditary graph

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Introduction. Given a tree, suppose that some of its vertices are labeled by sets which form a partition of $\{1, \ldots, n\}$, while its edges are weighted by some positive real numbers. Then let d_{ij} denote the sum of the weights of all the edges on the path connecting the vertices with i and j in their label sets. This results in a symmetric, $n \times n$ matrix $d = [d_{ij}]$, with zero diagonal, satisfying the *4-point condition,*

$$d_{ij} + d_{kl} \leq \max\{d_{ik} + d_{jl}, d_{il} + d_{jk}\}$$

for all i, j, k, l from $\{1, \ldots, n\}$. The converse, that this condition guarantees tree realizability, is also true (see, for example, Buneman [4]) and constitutes a well-known result used widely in taxonomy; cf. [1], [3] for pertinent references.

Furthermore, the weighted-tree representation for d is necessarily unique—provided that no redundant vertices are used, i.e., all vertices of degree less than 3 must be labeled. This mere uniqueness result of the representation also holds when arbitrary nonzero real weights are attached to the edges. This has recently been established by Hendy [6] using a novel technique based on Hadamard matrix transformations that also allows the recovery of the weighted tree, though in exponential time. Observe that a tree weighted with possibly negative reals still satisfies a relaxed 4-point condition, viz., at least two of the three distance sums are equal (and not necessarily the two larger ones). Our main result below will show that this condition characterizes tree realizability over $\mathbf{R}$. Interestingly, the distance matrix d of a graph G (unweighted, undirected) satisfies this relaxed 4-point condition exactly when G is a *distance-hereditary* graph, see [2]. Thus, we get a canonical tree representation for such graphs without extra effort.

The proof of the main theorem is inductive and can easily be adapted to construct the unique tree realization for d in polynomial time. Furthermore, our proof clarifies the respective roles played by the inequality and equality aspects of the classical 4-point condition (described above) in generating a tree representation—namely,

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previous proofs have exploited the inequality part in the construction of the tree; in fact, the equality part alone guarantees the existence and uniqueness of a tree representation with real edge weights, whereas the inequality part merely constrains the resulting edge weights to be positive. Also, in the classical situation, our proof can be further simplified, because in that case a certain complication cannot arise.

In order to cover both the classical case of tree metrics as well as the preceding one, we let the edge weights come from any submonoid of an abelian group without elements of order 2. Specifically, let Λ be an abelian monoid satisfying the following two conditions: for $\alpha, \beta, \xi, \zeta \in \Lambda$,

$$\alpha + \xi = \alpha + \zeta \text{ implies } \xi = \zeta \quad \text{(cancellation)},$$
$$\alpha + \alpha = \beta + \beta \text{ implies } \alpha = \beta \quad \text{(uniqueness of halves)}.$$

For $\alpha + \alpha$ we also write 2α, and in case $\alpha + \beta = 0$ is solvable, we write $\beta = -\alpha$. Canonical choices for Λ are $\Lambda = \mathbf{R}, \Lambda = \mathbf{R}^+, \Lambda = \frac{1}{2}\mathbf{Z}$, or $\Lambda = \frac{1}{2}\mathbf{N}$, under addition. The latter two choices are relevant when graphs are studied. One could, of course, also let $\Lambda = \mathbf{Z}_m$ (integers modulo m), where $m \geq 3$ is odd.

For a tree whose vertices are labeled as before and whose edges are weighted from Λ, the induced matrix d has the property that any four numbers, not necessarily distinct, chosen from $1, \ldots, n$ can be ordered as, say, i, j, k, l, so that

$$d_{ij} + d_{kl} + 2\xi = d_{ik} + d_{jl} = d_{il} + d_{kj} \quad \text{for some } \xi \in \Lambda.$$

We call this property the *4-point condition with respect to the monoid* Λ. In case $\Lambda = \mathbf{R}^+ \cup \{0\}$ (the nonnegative reals under addition), the 4-point condition with respect to Λ is nothing but the classical 4-point condition, which says that two of the three distance sums are equal and at least as large as the third. In case we choose $\Lambda = \mathbf{R}$, the condition simply requires that two of the distance sums are equal.

As in the classical case, we wish to realize a matrix over Λ satisfying the 4-point condition by a unique edge-weighted, labeled tree. We then require that Λ is cancellative and that half-elements are unique, and, in order to guarantee uniqueness, we must also insist that the tree has no unlabeled vertices of degree less than 3 and no zero edge weights.

THEOREM 1. *Let d be a symmetric, zero-diagonal, $n \times n$ matrix with entries in a cancellative abelian monoid Λ that has uniqueness of halves. Then d satisfies the 4-point condition with respect to Λ if and only if there exists a tree T that has no unlabeled vertices of degree less than 3 and that possesses a unique weighting of its edges by nonzero elements of Λ that realizes d. In this case, such a tree T is necessarily unique.*

Notice that both the cancellation and uniqueness of halves properties are essential hypotheses in Theorem 1. For instance, suppose that $\alpha + \xi = \alpha + \zeta$ holds in Λ for some $\xi \neq \zeta$. Then the two trees with three edges weighted α, α, ξ and α, α, ζ, respectively, yield the same matrix (of size 3). Similarly if $2\alpha = 2\beta$ but $\alpha \neq \beta$, then those two trees, now having edge weights α, α, β and β, β, α, respectively, give a common matrix. Even if we are willing to allow nonuniqueness of tree representations, the cancellation property alone is not sufficient to guarantee the existence of a tree representation for a matrix d satisfying the 4-point condition. The following proposition characterizes the extra condition required to guarantee the existence of tree realizations. For the sake of simplicity, we confine ourselves to abelian groups.

PROPOSITION 1. *Let Λ be an abelian group.*

(1) *Λ is Boolean (i.e., every nonzero element has order 2) if and only if every symmetric, zero-diagonal matrix d over Λ that satisfies the 4-point condition with respect to Λ can be realized on every tree for which $d_{ij} = 0$ whenever i, j are in the same label set.*

(2) *Λ has no elements of order 4 if and only if every such matrix d has a realization on at least one tree, with its edges weighted by elements of Λ.*

Note that in any tree realizations we consider (such as in the proofs), there is no loss of generality in assuming that all label sets are singletons, since we can restrict the domain of d to ensure this, and extend the resulting tree realization to one for d. Furthermore, with Λ as in Theorem 1, the extension is unique by the 4-point condition with respect to Λ.

Proof of Theorem 1. The "if" direction is clear. For the converse direction, we proceed by induction on the size of the matrix d, that is, the number of labels n. In the tree representations that follow, it is implicit that if an edge weight (described by some condition) is zero, then one collapses this edge, and the (possibly empty) sets of labels on the two ends of the edge are combined. For brevity, we sometimes speak of "vertex i" whenever that vertex is labeled by i.

For $n = 2$, there is nothing to show; if $d_{12} \neq 0$, one must simply assign the weight d_{12} to an edge whose ends are labeled 1 and 2.

For $n = 3$, consider the bush (i.e., a tree with no interior edges) having three leaves, labeled 1, 2, 3, each adjacent to a fourth, central vertex. Assign weights $\alpha, \beta, \gamma \in \Lambda$ to the edges of this tree incident with 1, 2, 3, respectively. Then

$$d_{23} + 2\alpha = d_{11} + d_{23} + 2\alpha = d_{12} + d_{13}$$

by virtue of the 4-point condition. Since Λ is cancellative and half-elements are unique (whenever they exist), this equation has a unique solution for α. Similarly, we describe β and γ uniquely. Then, for instance,

$$d_{23} + 2\alpha + 2\beta = d_{12} + d_{13} + 2\beta = 2d_{12} + d_{23},$$

from which we get $2(\alpha+\beta) = 2\alpha+2\beta = 2d_{12}$ (by cancellation) and hence $\alpha+\beta = d_{12}$, as desired. This settles the case $n = 3$.

As for $n = 4$, consider the generic binary tree consisting of four leaves, labeled $1, \ldots, 4$, with their incident edges weighted $\alpha_1, \ldots, \alpha_4$, respectively, together with a fifth edge, weighted ξ, which connects the path between 1 and 2 with that between 3 and 4. Thus for this tree, $d_{13} + d_{24} = d_{14} + d_{23}$. We claim that the α_i and ξ are uniquely defined. Indeed, applying the 4-point condition to 3-subsets, we obtain (cf., the case $n = 3$)

$$d_{i-1,i+1} + 2\alpha_i = d_{i-1,i} + d_{i,i+1},$$

where indices are taken modulo 4. This defines α_i uniquely. Moreover, the equation

$$d_{12} + d_{34} + 2\xi = d_{13} + d_{24}$$

yields a unique ξ for this labeled tree. Now,

$$d_{13} + d_{24} + 2\alpha_1 + 2\alpha_2 = d_{13} + d_{12} + d_{14} + 2\alpha_2 = 2d_{12} + d_{14} + d_{23},$$

from which we obtain $\alpha_1 + \alpha_2 = d_{12}$. Similarly, we obtain $\alpha_3 + \alpha_4 = d_{34}$. In order to recover the distance d_{14}, we compute

$$\begin{aligned} d_{13} + d_{24} + 2\alpha_1 + 2\xi + 2\alpha_4 &= d_{12} + d_{14} + d_{13} + 2\alpha_4 + 2\xi \\ &= d_{12} + d_{34} + 2\xi + 2d_{14} \\ &= d_{13} + d_{24} + 2d_{14}, \end{aligned}$$

yielding $\alpha_1 + \xi + \alpha_4 = d_{14}$. A similar result holds for d_{13}, d_{23}, d_{24}. In case $\xi = 0$, we obtain a bush. Notice that, by the cancellation and uniqueness of halves properties, as long as $\xi \neq 0$, no other labeling of the tree is consistent with the given matrix d. This proves the case $n = 4$.

Henceforth, let $n \geq 5$. Assume that every submatrix of d of size $n-1$ has a unique tree realization of the type claimed. Suppose first that for all distinct i, j, k, l, all three distance sums are equal. Then the lengths α_i and β_i of the edges incident with leaf i in the bushes connecting the triples i, j, k and i, j, l, respectively, satisfy

$$d_{jk} + 2\alpha_i = d_{ij} + d_{ik} \quad \text{and} \quad d_{ij} + d_{il} = d_{jl} + 2\beta_i.$$

Applying the hypothesized equality of distance sums and the properties of Λ to the sum of the preceding equations, we infer $\alpha_i = \beta_i$. This argument shows that the subbushes for all triples fit together consistently into a bush, and uniqueness of the weighted-tree representation follows as well.

So, assume that there exist two distinct sums; without loss of generality,

$$d_{14} + d_{23} = d_{12} + d_{34} \quad \text{but} \quad d_{13} + d_{24} \neq d_{12} + d_{34}.$$

Now consider the unique tree representations T_1, T_2, and $T_{1,2}$ of d restricted to $\{2, 3, \ldots, n\}, \{1, 3, 4, \ldots, n\}$, and $\{3, 4, \ldots, n\}$, respectively. Then $T_{1,2}$ is obtained from either tree T_1 or T_2 by deleting the labels 2 and 1, respectively, and "cleaning up" the resulting label-deleted trees. For example if $i \in \{1, 2\}$ labels a vertex v of degree at least three, we simply unlabel v. Otherwise, we delete v, and if v had degree one, we delete its incident edge, and if the other end of this edge v' has degree 2, we delete v'. In this last case or if v had degree 2, we then identify the two edges e_1 and e_2 incident with v', respectively, v, to give a new edge e. This edge is weighted by the sum of the weights of e_1 and e_2, unless this sum is zero, in which case e is contracted. Note that this contraction cannot occur in the classical situation. In order to recover T_1 or T_2 from $T_{1,2}$ we mark the edge or vertex of $T_{1,2}$ where the vertex labeled 1 or 2, respectively, is attached by a branch. A parent tree T for T_1 and T_2 is then obtained from T_{12} by reversing both of the processes that transformed T_1 and T_2 into T_{12}. However, we must show that this process and the corresponding edge weighting are well defined and unique when both marks (points of attachment of the 1-branch and the 2-branch) are either

(i) distinct but located in the interior of one and the same edge of $T_{1,2}$, or

(ii) coincident and located on a vertex.

In case (i), denote the end vertices of this edge by a and b. The vertices 3 and 4 belong to different components of $T_{1,2}$ minus the edge connecting a and b. Otherwise, say, if a is on paths from 3 and 4 to i in T_i for $i = 1, 2$, then either distance sum $d_{13} + d_{24}$ or $d_{14} + d_{23}$ would equal the sum of the edge weights along the paths from a to 3 and a to 4 in $T_{1,2}$, a to 2 in T_1, and a to 1 in T_2. This, however, conflicts with the hypothesis on the quartet 1, 2, 3, 4. Thus we may, without loss of generality, assume that either a is 3 or a lies between 3 and 1. In either case, subdivide the edge between a and b by two vertices c_1 and c_2, with c_1 being between a and c_2, so that c_i becomes the point of attachment of the i-branch ($i = 1, 2$). We must distinguish the four possible subcases:

$$(a \neq 3, b \neq 4), \quad (a = 3, b \neq 4), \quad (a \neq 3, b = 4), \quad \text{and} \quad (a = 3, b = 4).$$

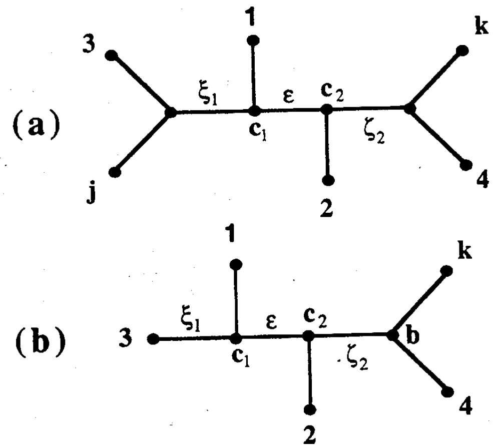

FIG. 1. *The first two subcases of case* (i) *in the proof of Theorem* 1.

For instance, the first two subcases are depicted in Fig. 1 (a) and (b), respectively. We will consider in detail only these two subcases—the treatment of the other two is similar.

Let ε be the unique element of Λ with

$$(*)\qquad d_{13} + d_{24} + 2\varepsilon = d_{14} + d_{23}.$$

Furthermore, there exist vertices labeled by j ($=3$ in subcase (b), and different from 3 in subcase (a)) and $k \neq 4$ such that the paths from j to 3 and k to 4 hit the edge between a and b only in a and b, respectively, and such that the following equalities hold:

$$(+)\qquad d_{i4} + d_{3j} + 2\xi_i = d_{i3} + d_{4j} \quad (i = 1, 2)$$

and

$$(++)\qquad d_{i3} + d_{4k} + 2\zeta_i = d_{i4} + d_{3k} \quad (i = 1, 2).$$

Now, adding up (*) and (+) for $i = 1$ yields

$$d_{24} + d_{3j} + 2(\xi_1 + \varepsilon) = d_{23} + d_{4j}.$$

Compared to equality (+) for $i = 2$, this implies

$$\xi_1 + \varepsilon = \xi_2$$

by the properties of the monoid Λ. Similarly, (++) for $i = 1$ compares to the sum of (*) and (++) for $i = 2$, thus yielding

$$\zeta_2 + \varepsilon = \zeta_1.$$

Therefore,

$$\xi_1 + \varepsilon + \zeta_2 = \xi_1 + \zeta_1 = \xi_2 + \zeta_2$$

is the weight of the edge between a and b (in $T_{1,2}$).

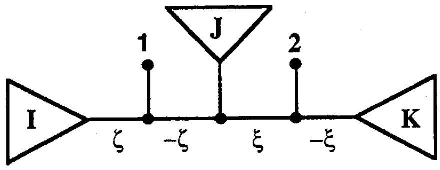

FIG. 2. *Case* (ii) *giving rise to compatible splits.*

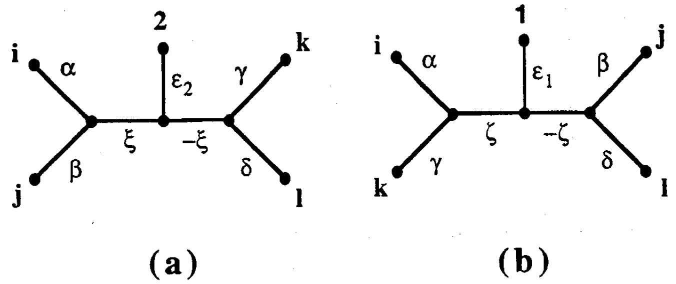

FIG. 3. *Case* (ii) *giving rise to incompatible splits.*

In case (ii) a problem may arise (though only in the nonclassical situation). Either expanded edge defines a split (i.e., a partition with two blocks) of $\{1,\ldots,n\}$. If the splits are compatible—that is, one is $\{I, J\cup K\}$ and the other is $\{I\cup J, K\}$—then we can still merge T_1 and T_2 uniquely into a single tree by the tree shown in Fig. 2, with the necessary $\pm\xi, \pm\zeta \in \Lambda$. However, the question remains of how to proceed if the splits were not compatible, that is, if for some i,j,k,l in $\{3,4,\ldots,n\}, T_1$ and T_2 contained respectively the subtrees in Fig. 3 (a) and (b).

We claim that this situation cannot arise because it is in conflict with the relation between the distance sums for the quartets $\{1,2,i,j\}, \{1,2,k,l\}, \{1,2,i,l\}$, and $\{1,2,j,k\}$. The three distance sums for each of these quartets are, respectively,

$$\begin{array}{lll}
d_{12}+\alpha+\beta, & \alpha+\zeta+\beta+\xi+\varepsilon_1+\varepsilon_2, & \beta-\zeta+\alpha+\xi+\varepsilon_1+\varepsilon_2,\\
d_{12}+\gamma+\delta, & \gamma+\zeta+\delta-\xi+\varepsilon_1+\varepsilon_2, & \delta-\zeta+\gamma-\xi+\varepsilon_1+\varepsilon_2,\\
d_{12}+\alpha+\delta, & \alpha+\zeta+\delta-\xi+\varepsilon_1+\varepsilon_2, & \delta-\zeta+\alpha+\xi+\varepsilon_1+\varepsilon_2,\\
d_{12}+\beta+\gamma, & \beta-\zeta+\gamma-\xi+\varepsilon_1+\varepsilon_2, & \gamma+\zeta+\beta+\xi+\varepsilon_1+\varepsilon_2.
\end{array}$$

Equality of the last two sums in row 1 or row 2 would give $2\zeta = 0$—that is, $\zeta = 0$—contrary to the hypothesis. Therefore, using cancellation, we infer that $d_{12} = \eta+\varepsilon_1+\varepsilon_2$, with

$$\eta \in \{\xi+\zeta, \xi-\zeta\} \cap \{-\xi+\zeta, -\xi-\zeta\}.$$

Since $\xi = 0$ or $\zeta = 0$ is impossible, we obtain either $\xi = \zeta$ or $\xi = -\zeta$, whence $d_{12} = \varepsilon_1+\varepsilon_2$ in either case, and thus any equality in row 3 gives $\xi = \zeta$. From the fourth row, we deduce, however, that $\zeta = -\xi$. This final contradiction completes the argument.

We conclude that T_1 and T_2 can indeed be combined to a unique tree T, in which all distances except possibly d_{12} are correctly represented. As for d_{12}, recall that $d_{14}+d_{23} = d_{12}+d_{34}$. Since d_{14}, d_{23}, and d_{34} have the correct values on T, so does d_{12} (by applying cancellation).

This completes the induction step and thus the proof.

Proof of Proposition 1. We may assume, without loss of generality, that the trees in question have all their leaves labeled. First we verify assertion (1). Suppose ε is an element of Λ such that $2\varepsilon \neq 0$. Then let $n = 4$ and consider the matrix d for which $d_{ij} = 0$ if $i + j$ is even and $d_{ij} = \varepsilon$ otherwise. This matrix has no realization on the tree with four leaves, for which the path joining leaves 1 and 2 is disjoint from the path joining leaves 3 and 4; cf. the case $n = 4$ in the proof of Theorem 1.

Conversely, assume that Λ is Boolean. Consider the bush S with leaves labeled $1, \ldots, n$. Given any fixed k, assign weight d_{ik} to the edge incident with leaf i ($i = 1, \ldots, n$). Since $2\varepsilon = 0$ for all $\varepsilon \in \Lambda$, we infer the equality $d_{ij} = d_{ik} + d_{jk}$ from the 4-point condition, so d is realized by S with this weighting. Now every tree T that contains vertices labeled $1, \ldots, n$ can be obtained from a subdivision T_0 of S by successively applying the following "edge swap" operation: given the tree T_k, so far constructed, realizing d, assume that some vertex x of T_k is connected to two vertices y and z by edges weighted α and β, respectively. Then remove the edge between x and z and create a new edge of weight $\alpha + \beta$ connecting y and z instead. The resulting tree T_{k+1} induces d as well. Eventually, we arrive at a subdivision T_m of T. Finally, contract all edges incident with unlabeled vertices of degree 2, which are not in T, and thereby add up the weights; this yields T.

As to (2), suppose Λ is an abelian group that contains an element ε of order 4. Define a 5×5 matrix d with entries in Λ by setting $d_{ij} = 2\varepsilon$ precisely if $\{i, j\} = \{2, 4\}, \{2, 3\}$, or $\{1, 4\}$ and setting $d_{ij} = 0$ otherwise. Then d satisfies the 4-point condition with respect to Λ. However, d has no tree representation with an edge weighting from Λ. This can be seen by restricting d to the sets $\{1, 2, 3, 4\}, \{1, 3, 4, 5\}$, and $\{1, 2, 3, 5\}$. Specifically, for $\{1, 2, 3, 4\}$,

$$d_{13} + d_{24} \neq d_{14} + d_{23} = d_{12} + d_{34},$$

and so, for any tree representation of d restricted to $\{1, 2, 3, 4\}$, the path joining the vertices labeled 1 and 3 must be disjoint from the path joining the vertices labeled 2 and 4. Similarly, by considering $\{1, 3, 4, 5\}$ and $\{1, 2, 3, 5\}$, we require that the path joining 3 and 5 is disjoint from the one joining 1 and 4 and that the path joining 1 and 5 is disjoint from the one joining 2 and 3, respectively. Clearly, however, these three constraints cannot be realized on a single tree, as claimed.

Conversely, suppose Λ has no element of order 4. Let Λ_0 denote the subgroup of Λ generated by the entries in d, together with one solution for the subsets $\{i, j, k, l\}$ of size at least 3 of the equation required of d by the 4-point condition for i, j, k, l. Since Λ_0 is finitely generated and has no element of order 4, the structure theorem for finitely generated abelian groups implies that there is an isomorphism ϕ: $\Lambda_0 \to \Gamma \times \Delta$, where Γ has no elements of order 2 and Δ is a Boolean group. Let d^Γ and d^Δ denote the projections of $\phi(d)$ onto Γ and Δ, respectively. Then d^Γ satisfies the 4-point condition according to Γ, and so, applying the previous theorem, there is a tree T and a weighting of its edges by nonzero elements of Γ that realizes d^Γ. We now "expand" T to allow for a tree representation for d^Δ. Specifically, for each vertex v of T that is assigned a set S of $s > 1$ labels, make v adjacent to s new leaves and assign each such leaf a unique label from S, thereby obtaining a tree T' having only singleton labels. Extend the previous edge weighting by Γ of T to T' by assigning weight $0 \in \Gamma$ to the new edges. Since d^Δ satisfies the 4-point condition with respect to Δ, part (1) shows that there is a weighting of the edges of T' by elements of Δ that realizes d^Δ. Now, for each edge e of T' let

$$\lambda(e) = \phi^{-1}(\gamma(e), \delta(e)),$$

where $\gamma(e) \in \Gamma$ and $\delta(e) \in \Delta$ are the weights that were previously assigned to e by considering d^Γ and d^Δ, respectively. Then $\lambda(e) \in \Lambda_0$, and the weighting of the edges of T' described by λ realizes d. This completes the proof.

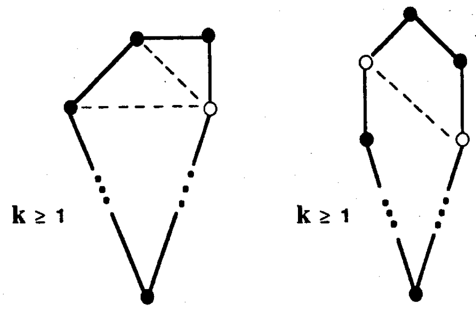

FIG. 4. *Configurations excluded by the 4-point condition with respect to* $\mathbf{R}$.

Distance-hereditary graphs. Let G be an unweighted, undirected, connected graph with vertices numbered 1 through n. The shortest-path metric d of G (which counts the edges in the shortest path connecting pairs of vertices in G) then takes values among $0, 1, \ldots, n-1$. There are several options for the abelian monoid Λ with respect to which the 4-point condition can be considered. If we take $\Lambda = \mathbf{N}$, then d satisfies this condition if and only if G is an unweighted tree. For $\Lambda = \frac{1}{2}\mathbf{N}$ the corresponding 4-point condition characterizes block graphs (graphs in which every maximal 2-connected subgraph (block) is complete); see Howorka [7]. The case $\Lambda = \frac{1}{2}\mathbf{Z}$ is more interesting, since it leads to a metric description of distance-hereditary graphs; see Bandelt and Mulder [2]. G is said to be *distance-hereditary* if every induced path (or subgraph) is isometric, that is, constitutes a subspace with respect to the metric d. Actually, we may compute distance modulo $2k+1$ for any $k \geq 1$ and arrive at the same class of graphs.

PROPOSITION 2. *A graph is distance-hereditary if and only if its shortest-path metric d satisfies the 4-point condition with respect to an abelian monoid Λ, where Λ may be chosen as $\Lambda = \frac{1}{2}\mathbf{Z}$ (or $\mathbf{R}$) or $\mathbf{Z}_m$ for $m \geq 3$ odd. Furthermore, G is bipartite and distance-hereditary if and only if its metric d satisfies the 4-point condition with respect to $\Lambda = \mathbf{Z}$.*

In view of Theorem 1, we can thus uniquely code a distance-hereditary graph G by a weighted tree over $\frac{1}{2}\mathbf{Z}$, where the vertices of the tree with degree smaller than three are labeled by the vertices of G. An immediate consequence of this is the following result (known to several people by now): the isomorphism problem for distance-hereditary graphs is easy. Indeed, two distance-hereditary graphs are isomorphic if and only if their associated weighted trees are isomorphic. For general graphs, by contrast, determining the complexity of the isomorphism question is a difficult and still unsolved problem [5]. Furthermore, the automorphism group of a distance-hereditary graph is isomorphic to that of a tree.

Proof of Proposition 2. According to [2, Thm. 2], G is distance-hereditary if and only if d satisfies the 4-point condition with respect to $\mathbf{R}$, so we only have to adjust for the expression of the distances modulo m. Assume that G is not distance-hereditary. Then there exists an isometric subgraph of the form shown in Fig. 4 possessing a cycle of length $2k+3$ or $2k+4$ ($k \geq 1$) with two (or one) possible chords as indicated by the dotted lines. The four shaded vertices in either cycle yield distance sums $k+1, k+2, k+3$ and $k+1, k+3, k+5$, respectively. In either case these are all different provided m is not 2 or 4.

Evidently, G is bipartite if and only if each distance sum

$$d_{ij} + d_{jk} + d_{ki}$$

is even. This is precisely the case when d satisfies the instances, for which $\#\{i, j, k, l\} = 3$, of the 4-point condition with respect to $\Lambda = \mathbf{Z}$. Thence the result.

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